THE MASS-CRITICAL NONLINEAR SCHRÖDINGER EQUATION WITH RADIAL DATA IN DIMENSIONS THREE AND HIGHER

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Abstract. We establish global well-posedness and scattering for solutions to the mass-critical nonlinear Schrödinger equation $iu_t + \Delta u = \pm |u|^{4/d} u$ for large spherically symmetric $L_x^2(\mathbb{R}^d)$ initial data in dimensions $d \geq 3$. In the focusing case we require that the mass is strictly less than that of the ground state. As a consequence, we obtain that in the focusing case, any spherically symmetric blowup solution must concentrate at least the mass of the ground state at the blowup time.

1. Introduction

The d-dimensional mass-critical nonlinear Schrödinger equation is given by

$$iu_t + \Delta u = F(u) \quad \text{with } F(u) := \mu |u|^{\frac{4}{d}} u \tag{1}$$

where u is a complex-valued function of spacetime $\mathbb{R} \times \mathbb{R}^d$. Here $\mu = \pm 1$, with $\mu = 1$ known as the defocusing equation and $\mu = -1$ as the focusing equation.

The name 'mass-critical' refers to the fact that the scaling symmetry

$$u(t,x) \mapsto u_\lambda(t,x) := \lambda^{-\frac{d}{2}} u(\lambda^{-2} t, \lambda^{-1} x) \tag{2}$$

leaves both the equation and the mass invariant. The mass of a solution is

$$M(u(t)) := \int_{\mathbb{R}^d} |u(t,x)|^2 \, dx \tag{3}$$

and is conserved under the flow.

In this paper, we investigate the Cauchy problem for (1) for spherically symmetric $L_x^2(\mathbb{R}^d)$ initial data in dimensions $d \geq 3$ by adapting the recent argument from [26], which treated the case $d = 2$. Before describing our results, we need to review some background material. We begin by making the notion of a solution more precise:

Definition 1.1 (Solution)**.** A function $u : I \times \mathbb{R}^d \to \mathbb{C}$ on a non-empty time interval $I \subset \mathbb{R}$ is a *solution* (more precisely, a strong $L_x^2(\mathbb{R}^d)$ solution) to (1) if it lies in the class $C_t^0 L_x^2(K \times \mathbb{R}^d) \cap L_{t,x}^{2(d+2)/d}(K \times \mathbb{R}^d)$ for all compact $K \subset I$, and obeys the Duhamel formula

$$u(t_1) = e^{i(t_1 - t_0)\Delta} u(t_0) - i \int_{t_0}^{t_1} e^{i(t_1 - t)\Delta} F(u(t)) \, dt \tag{4}$$

for all $t_0, t_1 \in I$. Note that by Lemma 2.7 below, the condition $u \in L_{t,x}^{2(d+2)/d}$ locally in time guarantees that the integral converges, at least in a weak-L_x^2 sense.

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Remark. The condition that u is in $L_{t,x}^{2(d+2)/d}$ locally in time is natural. This space appears in the Strichartz inequality (Lemma 2.7); consequently, all solutions to the linear problem lie in this space. Existence of solutions to (1) in this space is guaranteed by the local theory discussed below; it is also necessary in order to ensure uniqueness of solutions in this local theory. Solutions to (1) in this class have been intensively studied, see for example [1, 4, 8, 9, 10, 25, 30, 38, 39, 40, 41].

Associated to this notion of solution is a corresponding notion of blowup. As we will see in Theorem 1.3 below, this precisely corresponds to the impossibility of continuing the solution.

Definition 1.2 (Blowup)**.** We say that a solution u to (1) *blows up forward in time* if there exists a time $t_0 \in I$ such that

$$\int_{t_0}^{\sup I} \int_{\mathbb{R}^d} |u(t,x)|^{2(d+2)/d}\,dx\,dt = \infty$$

and that u *blows up backward in time* if there exists a time $t_0 \in I$ such that

$$\int_{\inf I}^{t_0} \int_{\mathbb{R}^d} |u(t,x)|^{2(d+2)/d}\,dx\,dt = \infty.$$

The local theory for (1) was worked out by Cazenave and Weissler [9]. They constructed local-in-time solutions for arbitrary initial data in $L_x^2(\mathbb{R}^d)$; however, due to the critical nature of the equation, the resulting time of existence depends on the profile of the initial data and not merely on its L_x^2-norm. Cazenave and Weissler also constructed global solutions for small initial data. We summarize their results in the theorem below.

Theorem 1.3 (Local well-posedness, [9, 10])**.** *Given $u_0 \in L_x^2(\mathbb{R}^d)$ and $t_0 \in \mathbb{R}$, there exists a unique maximal-lifespan solution u to (1) with $u(t_0) = u_0$. We will write I for the maximal lifespan. This solution also has the following properties:*

- *(Local existence) I is an open neighbourhood of t_0.*
- *(Mass conservation) The solution u obeys mass conservation: $M(u(t)) = M(u_0)$ for all $t \in I$.*
- *(Blowup criterion) If $\sup(I)$ or $\inf(I)$ is finite, then u blows up in the corresponding time direction.*
- *(Continuous dependence) The map that takes initial data to the corresponding strong solution is uniformly continuous on compact time intervals for bounded sets of initial data.*
- *(Scattering) If $\sup(I) = +\infty$ and u does not blow up forward in time, then u scatters forward in time, that is, there exists a unique $u_+ \in L_x^2(\mathbb{R}^d)$ such that*

$$\lim_{t\to+\infty} \|u(t) - e^{it\Delta}u_+\|_{L_x^2(\mathbb{R}^d)} = 0.$$

  Similarly, if $\inf(I) = -\infty$ and u does not blow up backward in time, then u scatters backward in time, that is, there is a unique $u_- \in L_x^2(\mathbb{R}^d)$ so that

$$\lim_{t\to-\infty} \|u(t) - e^{it\Delta}u_-\|_{L_x^2(\mathbb{R}^d)} = 0.$$

- *(Small data global existence) If $M(u_0)$ is sufficiently small depending on d, then u is a global solution with finite $L_{t,x}^{2(d+2)/d}$ norm.*

It is widely believed that in the defocusing case, all L_x^2 initial data lead to a global solution with finite $L_{t,x}^{2(d+2)/d}$ spacetime norm (and hence also scattering).

In the focusing case, the general consensus is more subtle. Let Q denote the *ground state*, that is, the unique positive radial solution to

$$\Delta Q + Q^{1+4/d} = Q. \tag{5}$$

(The existence and uniqueness of Q was established in [2] and [27] respectively.) Then $u(t,x) := e^{it}Q(x)$ is a solution to (1), which is global but blows up both forward and backward in time (in the sense of Definition 1.2). More dramatically, by applying the pseudoconformal transformation to u, we obtain a solution

$$v(t,x) := |t|^{-d/2} e^{i\frac{|x|^2-4}{4t}} Q\Big(\frac{x}{t}\Big) \tag{6}$$

with the same mass that blows up in finite time. It is widely believed that this ground state example is the minimal-mass obstruction to global well-posedness and scattering in the focusing case.

To summarize, we subscribe to

Conjecture 1.4 (Global existence and scattering)**.** *Let $d \geq 1$ and $\mu = \pm 1$. In the defocusing case $\mu = +1$, all maximal-lifespan solutions to* (1) *are global and do not blow up either forward or backward in time. In the focusing case $\mu = -1$, all maximal-lifespan solutions u to* (1) *with $M(u) < M(Q)$ are global and do not blow up either forward or backward in time.*

Remark. While this conjecture is phrased for $L_x^2(\mathbb{R}^d)$ solutions, it is equivalent to a scattering claim for smooth solutions; see [1, 7, 25, 38]. In [3, 38], it is also shown that the global existence and the scattering claims are equivalent in the $L_x^2(\mathbb{R}^d)$ category.

The contribution of this paper toward settling this conjecture is

Theorem 1.5. *Let $d \geq 3$. Then Conjecture 1.4 is true in the class of spherically symmetric initial data (for either choice of sign μ).*

Conjecture 1.4 has been the focus of much intensive study and several partial results for various choices of d, μ, and sometimes with the additional assumption of spherical symmetry. The most compelling evidence in favour of this conjecture stems from results obtained under the assumption that u_0 has additional regularity. For the defocusing equation, it is easy to prove global well-posedness for initial data in H_x^1; this follows from the usual contraction mapping argument combined with the conservation of mass and energy; see, for example, [10]. Recall that the energy is given by

$$E(u(t)) := \int_{\mathbb{R}^d} \frac{1}{2}|\nabla u(t,x)|^2 + \mu \frac{d}{2(d+2)} |u(t,x)|^{\frac{2(d+2)}{d}}\, dx. \tag{7}$$

Note that for general L_x^2 initial data, the energy need not be finite.

The focusing equation with data in H_x^1 was treated by Weinstein [46]. A key ingredient was his proof of the sharp Gagliardo–Nirenberg inequality:

Theorem 1.6 (Sharp Gagliardo–Nirenberg, [46])**.**

$$\int_{\mathbb{R}^d} \big|f(x)\big|^{\frac{2(d+2)}{d}}\, dx \leq \frac{d+2}{d} \bigg(\frac{\|f\|_{L^2}^2}{\|Q\|_{L^2}^2}\bigg)^{\frac{2}{d}} \int_{\mathbb{R}^d} \big|\nabla f(x)\big|^2\, dx. \tag{8}$$

As noticed by Weinstein, this inequality implies that the energy (7) is positive once $M(u_0) < M(Q)$; indeed, it gives an upper bound on the $\dot H^1_x$-norm of the solution at all times of existence. Combining this with a contraction mapping argument and the conservation of mass and energy, Weinstein proved global well-posedness for the focusing equation with initial data in H^1_x and mass smaller than that of the ground state.

Note that the iterative procedure used to obtain a global solution both for the defocusing and the focusing equations with initial data in H^1_x does not yield finite spacetime norms; in particular, scattering does not follow even for more regular initial data.

In dimensions one and two, there has been much work [4, 12, 13, 15, 16, 18, 21, 42] devoted to lowering the regularity of the initial data from H^1_x toward $L^2_x(\mathbb{R}^d)$ and thus, toward establishing the conjecture. For analogous results in higher dimensions, see [17, 45].

In the case of spherically symmetric solutions, Conjecture 1.4 was recently settled in the high-dimensional defocusing case $\mu = +1$, $d \geq 3$ in [40]; thus only the $\mu = -1$ case of Theorem 1.5 is new. However, the techniques used in [40] do not seem to be applicable to the focusing problem, primarily because the Morawetz inequality is no longer coercive in that case. Instead, our argument is based on the recent preprint [26], which resolved the conjecture for $\mu = \pm 1$, $d = 2$, and spherically symmetric data. In turn, [26] uses techniques developed to treat the analogous conjecture for the energy-critical problem, such as [5, 14, 32, 37, 43, 44] and particularly [23]. We will give a more thorough discussion of the relation of the current work to these predecessors later, when we outline the argument.

1.1. **Mass concentration in the focusing problem.** Neither Theorem 1.5 nor Conjecture 1.4 address the focusing problem for masses greater than or equal to that of the ground state. In this case, blowup solutions exist and attention has been focused on describing their properties. For instance, finite-time blowup solutions with finite energy and mass equal to that of the ground state have been completely characterized by Merle [28]; they are precisely the ground state solution up to symmetries of the equation.

Several works have shown that finite-time blowup solutions must concentrate a positive amount of mass around the blowup time T^*. For finite energy data, see [29, 31, 47] where it is shown that there exists $x(t) \in \mathbb{R}^d$ so that

$$\liminf_{t \nearrow T^*} \int_{|x-x(t)|\leq R} \bigl|u(t,x)\bigr|^2\, dx \geq M(Q)$$

for any $R > 0$. For merely $L^2_x(\mathbb{R}^2)$ initial data, Bourgain [4] proved that some small amount of mass must concentrate in parabolic windows (at least along a subsequence):

$$\limsup_{t \nearrow T^*} \sup_{x_0 \in \mathbb{R}^2} \int_{|x-x_0|\leq (T^*-t)^{1/2}} \bigl|u(t,x)\bigr|^2\, dx \geq c,$$

where c is a small constant depending on the mass of u. This result was extended to other dimensions in [1, 25].

Combining Theorem 1.5 with the argument in [26, §10], one obtains the following concentration result.

Corollary 1.7 (Blowup solutions concentrate the mass of the ground state)**.** *Let $d \geq 3$ and $\mu = -1$. Let u be a spherically symmetric solution to* (1) *that blows up at time $0 < T^* \leq \infty$. If $T^* < \infty$, then there exists a sequence $t_n \nearrow T^*$ so that for any sequence $R_n \in (0, \infty)$ obeying $(T^* - t_n)^{-1/2} R_n \to \infty$,*

$$\limsup_{n\to\infty} \int_{|x|\leq R_n} |u(t_n, x)|^2 \, dx \geq M(Q). \tag{9}$$

If $T^* = \infty$, then there exists a sequence $t_n \to \infty$ such that for any sequence $R_n \in (0, \infty)$ with $t_n^{-1/2} R_n \to \infty$,

$$\limsup_{n\to\infty} \int_{|x|\leq R_n} |u(t_n, x)|^2 \, dx \geq M(Q). \tag{10}$$

The analogous statement holds in the negative time direction.

1.2. **Outline of the proof.** Beginning with Bourgain's seminal work [5] on the energy-critical NLS, it has become apparent that in order to prove spacetime bounds for general solutions, it is sufficient to treat a special class of solutions, namely, those that are simultaneously localized in both frequency and space. For further developments, see [14, 32, 37, 43, 44].

A new and much more efficient alternative to Bourgain's induction on mass (or energy) method has recently been developed. It uses a (concentration) compactness technique to isolate minimal-mass/energy blowup solutions as opposed to the almost-blowup solutions of the induction method. Building on earlier developments in [1, 4, 24, 25, 30], Kenig and Merle [23] used this method to treat the energy-critical focusing problem with radial data in dimensions three, four, and five.

To explain what the concentration compactness argument gives in our context, we need to introduce the following important notion:

Definition 1.8 (Almost periodicity modulo scaling)**.** Given $d \geq 1$ and $\mu = \pm 1$, a solution u with lifespan I is said to be *almost periodic modulo scaling* if there exists a (possibly discontinuous) function $N : I \to \mathbb{R}^+$ and a function $C : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\int_{|x|\geq C(\eta)/N(t)} |u(t, x)|^2 \ dx \leq \eta$$

and

$$\int_{|\xi|\geq C(\eta)N(t)} |\hat{u}(t, \xi)|^2 \ d\xi \leq \eta$$

for all $t \in I$ and $\eta > 0$. We refer to the function N as the *frequency scale function* and to C as the *compactness modulus function*.

Remarks. 1. The parameter $N(t)$ measures the frequency scale of the solution at time t, and $1/N(t)$ measures the spatial scale; see [39, 40] for further discussion. Note that we have the freedom to modify $N(t)$ by any bounded function of t, provided that we also modify the compactness modulus function C accordingly. In particular, one could restrict $N(t)$ to be a power of 2 if one wished, although we will not do so here. Alternatively, the fact that the solution trajectory $t \mapsto u(t)$ is continuous in $L^2_x(\mathbb{R}^d)$ can be used to show that the function N may be chosen to depend continuously on t.

2. By the Ascoli–Arzela Theorem, a family of functions is precompact in $L^2_x(\mathbb{R}^d)$ if and only if it is norm-bounded and there exists a compactness modulus function

C so that

$$\int_{|x|\geq C(\eta)} |f(x)|^2 \, dx + \int_{|\xi|\geq C(\eta)} |\hat{f}(\xi)|^2 \, d\xi \leq \eta$$

for all functions f in the family. Thus, an equivalent formulation of Definition 1.8 is as follows: u is almost periodic modulo scaling if and only if

$$\{u(t) : t \in I\} \subseteq \{f(x/\lambda) : \lambda \in (0,\infty) \text{ and } f \in K\}.$$

for some compact subset K of $L^2_x(\mathbb{R}^d)$.

In [39, Theorems 1.13 and 7.2] the following result was established (see also [1, 25]), showing that any failure of Conjecture 1.4 must be 'caused' by a very special type of solution. For simplicity we state it only in the spherically symmetric case.

Theorem 1.9 (Reduction to almost periodic solutions)**.** *Fix μ and $d \geq 2$. Suppose that Conjecture 1.4 fails for spherically symmetric data. Then, there exists a spherically symmetric maximal-lifespan solution u which is almost periodic modulo scaling and which blows up both forward and backward in time, and in the focusing case we also have $M(u) < M(Q)$.*

In [26], this result was further refined so as to identify three specific enemies. Once again, we state it only in the spherically symmetric case.

Theorem 1.10 (Three special scenarios for blowup, [26])**.** *Fix μ and $d \geq 2$ and suppose that Conjecture 1.4 fails for spherically symmetric data. Then there exists a spherically symmetric maximal-lifespan solution u which is almost periodic modulo scaling, blows up both forward and backward in time, and in the focusing case also obeys $M(u) < M(Q)$. Moreover, the solution u may be chosen to match one of the following three scenarios:*

- *(Soliton-like solution) We have $I = \mathbb{R}$ and*
$$N(t) = 1 \tag{11}$$
for all $t \in \mathbb{R}$ (thus the solution stays in a bounded space/frequency range for all time).
- *(Double high-to-low frequency cascade) We have $I = \mathbb{R}$,*
$$\liminf_{t\to-\infty} N(t) = \liminf_{t\to+\infty} N(t) = 0, \tag{12}$$
and
$$\sup_{t\in\mathbb{R}} N(t) < \infty \tag{13}$$
for all $t \in I$.
- *(Self-similar solution) We have $I = (0, +\infty)$ and*
$$N(t) = t^{-1/2} \tag{14}$$
for all $t \in I$.

In light of this result, the proof of Theorem 1.5 is reduced to showing that none of these three scenarios can occur. In doing this, we follow the model set forth in [26]. In all cases, the key step is to prove that u has additional regularity. Indeed, to treat the first two scenarios, we need more than one derivative in L^2_x; for the self-similar scenario, H^1_x suffices. The possibility of showing such additional regularity

stems from the fact that u is both frequency and space localized; this in turn is an expression of the fact that u has minimal mass among all blowup solutions.

A further manifestation of this minimality is the absence of a scattered wave at the endpoints of the lifespan I; more formally, we have

Lemma 1.11 ([39, Section 6])**.** *Let u be a solution to* (1) *which is almost periodic modulo scaling on its maximal-lifespan I. Then, for all $t \in I$,*

$$\begin{aligned} u(t) &= \lim_{T \nearrow \sup I} i \int_t^T e^{i(t-t')\Delta} F(u(t'))\, dt' \\ &= - \lim_{T \searrow \inf I} i \int_T^t e^{i(t-t')\Delta} F(u(t'))\, dt', \end{aligned} \tag{15}$$

as weak limits in L^2_x.

Another important property of solutions that are almost periodic modulo scaling is that the behaviour of the spacetime norm is governed by that of $N(t)$. More precisely, we have the following lemma from [26]:

Lemma 1.12 (Spacetime bound, [26])**.** *Let u be a non-zero solution to* (1) *with lifespan I, which is almost periodic modulo scaling with frequency scale function $N : I \to \mathbb{R}^+$. If J is any subinterval of I, then*

$$\int_J N(t)^2\, dt \lesssim_u \int_J \int_{\mathbb{R}^d} |u(t,x)|^{\frac{2(d+2)}{d}}\, dx\, dt \lesssim_u 1 + \int_J N(t)^2\, dt.$$

The nonexistence of self-similar solutions is proved in Section 3. We first prove that any such solution would belong to $C^0_t H^1_x$ and then observe that H^1_x solutions are global (see the discussion after Theorem 1.5), while self-similar solutions are not.

For the remaining two cases, higher regularity is proved in Section 5. In order to best take advantage of Lemma 1.11, we exploit a decomposition of spherically symmetric functions into incoming and outgoing waves; this is discussed in Section 4.

In Section 6, we use the additional regularity together with the conservation of energy to preclude the double high-to-low frequency cascade. In Section 7, we disprove the existence of soliton-like solutions using a truncated virial identity in much the same manner as [23].

As noted earlier, the argument just described is closely modelled on [26], which treated the same equation in two dimensions. The main obstacle in extending that work to higher dimensions is the fractional power appearing in the nonlinearity. This problem presents itself when we prove additional regularity, which is already the most demanding part of [26]. Additional regularity is proved via a bootstrap argument using Duhamel's formula. However, fractional powers can downgrade regularity (a fractional power of a smooth function need not be smooth); in particular, they preclude the simple Littlewood-Paley arithmetic that is usually used in the case of polynomial nonlinearities.

The remedy is twofold: first we use fractional chain rules (see Lemmas 2.3 and 2.4) that allow us to take more than one derivative of a nonlinearity that is merely $C^{1+\frac{4}{d}}$ in u. Secondly, we push through the resulting complexities in the bootstrap argument. An important role is played by Lemma 2.1 (a Gronwall-type result), which we use to untangle the intricate relationship between frequencies in u and those in $|u|^{\frac{4}{d}} u$.

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2. Notation and linear estimates

This section contains the basic linear estimates we use repeatedly in the paper.

2.1. **Some notation.**

We use $X \lesssim Y$ or $Y \gtrsim X$ whenever $X \leq CY$ for some constant $C > 0$. We use $O(Y)$ to denote any quantity X such that $|X| \lesssim Y$. We use the notation $X \sim Y$ whenever $X \lesssim Y \lesssim X$. The fact that these constants depend upon the dimension d will be suppressed. If C depends upon some additional parameters, we will indicate this with subscripts; for example, $X \lesssim_u Y$ denotes the assertion that $X \leq C_u Y$ for some C_u depending on u.

We use the 'Japanese bracket' convention $\langle x\rangle := (1 + |x|^2)^{1/2}$.

We write $L_t^q L_x^r$ to denote the Banach space with norm

$$\|u\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^d)} := \Big(\int_{\mathbb{R}} \Big(\int_{\mathbb{R}^d} |u(t,x)|^r \, dx\Big)^{q/r} dt\Big)^{1/q},$$

with the usual modifications when q or r are equal to infinity, or when the domain $\mathbb{R}\times\mathbb{R}^d$ is replaced by a smaller region of spacetime such as $I \times \mathbb{R}^d$. When $q = r$ we abbreviate $L_t^q L_x^q$ as $L_{t,x}^q$.

The next lemma is a variant of Gronwall's inequality that we will use to handle some bootstrap arguments below. The proof given is a standard application of techniques from the theories of Volterra and Toeplitz operators.

Lemma 2.1 (A Gronwall inequality)**.** *Fix $r \in (0,1)$ and $K \geq 4$. Let b_k be a bounded sequence of non-negative numbers and x_k a sequence obeying $0 \leq x_k \leq b_k$ for $0 \leq k < K$ and*

$$0 \leq x_k \leq b_k + \sum_{l=0}^{k-K} r^{k-l} x_l \qquad \text{for all } k \geq K. \tag{16}$$

Then

$$0 \leq x_k \lesssim \sum_{l=0}^{k} r^{k-l} \exp\big\{\tfrac{\log(K-1)}{K-1}\,(k-l)\big\} b_l \tag{17}$$

for all $k \geq 0$. In particular, if $b_k = O(2^{-k\sigma})$ and $2^\sigma r(K-1)^{1/(K-1)} < 1$, then $x_k = O(2^{-k\sigma})$.

Proof. Elementary arguments show that we need only obtain the bound for the case of equality, namely, where

$$(1 - A)x = b. \tag{18}$$

Here x and b denote the semi-infinite vectors built from the corresponding sequences, while A is the matrix with entries

$$A_{k,l} = \begin{cases} r^{k-l} & : \text{if } k-l \geq K, \\ 0 & : \text{otherwise.} \end{cases}$$

The triangular structure of A guarantees that (18) can be solved (though not *a priori* in ℓ^∞); more precisely, it guarantees that the geometric series for $(1-A)^{-1}$ converges entry-wise. To obtain bounds for the entries of this inverse matrix, it is simplest to use a functional model: under the mapping of sequences to functions

$$x_k \mapsto \sum_{k=0}^{\infty} x_k z^k \quad \text{and} \quad b_k \mapsto \sum_{k=0}^{\infty} b_k z^k,$$

the matrix A becomes multiplication by $r^K z^K (1-rz)^{-1}$. In the same way, the entries of $(1-A)^{-1}$ come from the Taylor coefficients of

$$a(z) := \frac{1-rz}{1-rz-r^K z^K}.$$

Using $e^x \geq 1+x$ with $x = -\log|rz|$, we see that

$$|1-rz| \geq \big(\tfrac{1}{r|z|} - 1\big) r|z| \geq \frac{\log(K-1)}{K-1} r|z| \geq \log(K-1)\, r^K |z|^K$$

on the disk $|z| \leq r^{-1}(K-1)^{-1/(K-1)}$. This shows that $a(z)$ is bounded and analytic on this disk. (Note that the hypothesis $K \geq 4$ implies that $\log(K-1) > 1$.) The inequality (17) now follows from the standard Cauchy estimates. $\square$

2.2. **Basic harmonic analysis.** Let $\varphi(\xi)$ be a radial bump function supported in the ball $\{\xi \in \mathbb{R}^d : |\xi| \leq \frac{11}{10}\}$ and equal to 1 on the ball $\{\xi \in \mathbb{R}^d : |\xi| \leq 1\}$. For each number $N > 0$, we define the Fourier multipliers

$$\begin{aligned} \widehat{P_{\leq N} f}(\xi) &:= \varphi(\xi/N)\hat{f}(\xi) \\ \widehat{P_{> N} f}(\xi) &:= (1-\varphi(\xi/N))\hat{f}(\xi) \\ \widehat{P_N f}(\xi) &:= \psi(\xi/N)\hat{f}(\xi) := (\varphi(\xi/N) - \varphi(2\xi/N))\hat{f}(\xi) \end{aligned}$$

and similarly $P_{<N}$ and $P_{\geq N}$. We also define

$$P_{M<\cdot\leq N} := P_{\leq N} - P_{\leq M} = \sum_{M<N'\leq N} P_{N'}$$

whenever $M < N$. We will usually use these multipliers when M and N are *dyadic numbers* (that is, of the form 2^n for some integer n); in particular, all summations over N or M are understood to be over dyadic numbers. Nevertheless, it will occasionally be convenient to allow M and N to not be a power of 2. Note that P_N is not truly a projection; to get around this, we will occasionally need to use fattened Littlewood-Paley operators:

$$\tilde{P}_N := P_{N/2} + P_N + P_{2N}. \tag{19}$$

These obey $P_N \tilde{P}_N = \tilde{P}_N P_N = P_N$.

As with all Fourier multipliers, the Littlewood-Paley operators commute with the propagator $e^{it\Delta}$, as well as with differential operators such as $i\partial_t + \Delta$. We will use basic properties of these operators many many times, including

Lemma 2.2 (Bernstein estimates)**.** *For $1 \leq p \leq q \leq \infty$,*

$$\big\| |\nabla|^{\pm s} P_N f \big\|_{L_x^p(\mathbb{R}^d)} \sim N^{\pm s} \|P_N f\|_{L_x^p(\mathbb{R}^d)},$$
$$\|P_{\leq N} f\|_{L_x^q(\mathbb{R}^d)} \lesssim N^{\frac{d}{p}-\frac{d}{q}} \|P_{\leq N} f\|_{L_x^p(\mathbb{R}^d)},$$
$$\|P_N f\|_{L_x^q(\mathbb{R}^d)} \lesssim N^{\frac{d}{p}-\frac{d}{q}} \|P_N f\|_{L_x^p(\mathbb{R}^d)}.$$

The next few results provide important tools for dealing with the fractional power appearing in the nonlinearity.

Lemma 2.3 (Fractional chain rule for a C^1 function, [11])**.** *Suppose $G \in C^1(\mathbb{C})$, $s \in (0,1]$, and $1 < p, p_1, p_2 < \infty$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Then,*

$$\| |\nabla|^s G(u)\|_p \lesssim \|G'(u)\|_{p_1} \| |\nabla|^s u\|_{p_2}.$$

When the function G is no longer C^1, but merely Hölder continuous, we have the following useful chain rule:

Lemma 2.4 (Fractional chain rule for a Hölder continuous function, [43])**.** *Let G be a Hölder continuous function of order $0 < \alpha < 1$. Then, for every $0 < s < \alpha$, $1 < p < \infty$, and $\frac{s}{\alpha} < \sigma < 1$ we have*

$$\big\| |\nabla|^s G(u)\big\|_p \lesssim \big\| |u|^{\alpha - \frac{s}{\sigma}} \big\|_{p_1} \big\| |\nabla|^\sigma u\big\|^{\frac{s}{\sigma}}_{\frac{s}{\sigma} p_2}, \tag{20}$$

provided $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $(1 - \frac{s}{\alpha\sigma}) p_1 > 1$.

Corollary 2.5. *Let $0 \leq s < 1 + \frac{4}{d}$. Then, on any spacetime slab $I \times \mathbb{R}^d$ we have*

$$\big\| |\nabla|^s F(u)\big\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} \lesssim \big\| |\nabla|^s u\big\|_{L_{t,x}^{\frac{2(d+2)}{d}}} \|u\|^{\frac{4}{d}}_{L_{t,x}^{\frac{2(d+2)}{d}}}$$

and

$$\big\| |\nabla|^s F(u)\big\|_{L_t^\infty L_x^{\frac{2r}{r+4}}} \lesssim \big\| |\nabla|^s u\big\|_{L_t^\infty L_x^2} \|u\|^{\frac{4}{d}}_{L_t^\infty L_x^{\frac{2r}{d}}},$$

for any $\max\{d, 4\} \leq r \leq \infty$.

Proof. Fix a compact interval I. Throughout the proof, all spacetime estimates will be on $I \times \mathbb{R}^d$.

We begin with the first claim. For $0 < s \leq 1$, this is an easy consequence of Lemma 2.3. We now address the case $1 < s < 1 + \frac{4}{d}$. By the chain rule and the fractional product rule, we estimate

$$\begin{aligned}
\big\| |\nabla|^s F(u)\big\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} &\lesssim \big\| |\nabla|^{s-1} \big(\nabla u F_z(u) + \nabla \bar{u} F_{\bar{z}}(u)\big)\big\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} \\
&\lesssim \big\| |\nabla|^s u\big\|_{L_{t,x}^{\frac{2(d+2)}{d}}} \|u\|^{\frac{4}{d}}_{L_{t,x}^{\frac{2(d+2)}{d}}} \\
&\quad + \|\nabla u\|_{L_{t,x}^{\frac{2(d+2)}{d}}} \Big[\big\| |\nabla|^{s-1} F_z(u)\big\|_{L_{t,x}^{\frac{d+2}{2}}} + \big\| |\nabla|^{s-1} F_{\bar{z}}(u)\big\|_{L_{t,x}^{\frac{d+2}{2}}} \Big].
\end{aligned}$$

The claim will follow from this, once we establish

$$\big\| |\nabla|^{s-1} F_z(u)\big\|_{L_{t,x}^{\frac{d+2}{2}}} + \big\| |\nabla|^{s-1} F_{\bar{z}}(u)\big\|_{L_{t,x}^{\frac{d+2}{2}}} \lesssim \big\| |\nabla|^\sigma u\big\|^{\frac{s-1}{\sigma}}_{L_{t,x}^{\frac{2(d+2)}{d}}} \|u\|^{\frac{4}{d} - \frac{s-1}{\sigma}}_{L_{t,x}^{\frac{2(d+2)}{d}}} \tag{21}$$

for some $\frac{d(s-1)}{4} < \sigma < 1$. Indeed, one simply has to note that by interpolation,

$$\big\||\nabla|^\sigma u\big\|_{L_{t,x}^{\frac{2(d+2)}{d}}} \lesssim \big\||\nabla|^s u\big\|_{L_{t,x}^{\frac{2(d+2)}{d}}}^{\frac{\sigma}{s}} \|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}}^{1-\frac{\sigma}{s}}$$

and

$$\|\nabla u\|_{L_{t,x}^{\frac{2(d+2)}{d}}} \lesssim \big\||\nabla|^s u\big\|_{L_{t,x}^{\frac{2(d+2)}{d}}}^{\frac{1}{s}} \|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}}^{1-\frac{1}{s}}.$$

To derive (21), we remark that F_z and $F_{\bar z}$ are Hölder continuous functions of order $\frac{4}{d}$ and use Lemma 2.5 (with $\alpha := \frac{4}{d}$ and $s := s-1$).

We now turn to the second claim. Note that the condition $r \geq 4$ simply insures that $\frac{2r}{r+4} \geq 1$. For $0 < s \leq 1$, the claim follows immediately from Lemma 2.3. Let us consider the case $1 < s < 1 + \frac{4}{d}$. By the chain rule and the fractional product rule,

$$\begin{aligned}
\big\||\nabla|^s F(u)\big\|_{L_t^\infty L_x^{\frac{2r}{r+4}}} &\lesssim \big\||\nabla|^{s-1}\big(\nabla u F_z(u) + \nabla \bar u F_{\bar z}(u)\big)\big\|_{L_t^\infty L_x^{\frac{2r}{r+4}}} \\
&\lesssim \big\||\nabla|^s u\big\|_{L_t^\infty L_x^2} \|u\|_{L_t^\infty L_x^{\frac{2r}{d}}}^{\frac{4}{d}} \\
&\quad + \|\nabla u\|_{L_t^\infty L_x^{\frac{2rs}{r+(s-1)d}}} \big\||\nabla|^{s-1} O\big(|u|^{\frac{4}{d}}\big)\big\|_{L_t^\infty L_x^{\frac{2rs}{(r-d)(s-1)+4s}}}.
\end{aligned}$$

By interpolation,

$$\|\nabla u\|_{L_t^\infty L_x^{\frac{2rs}{r+(s-1)d}}} \lesssim \big\||\nabla|^s u\big\|_{L_t^\infty L_x^2}^{\frac{1}{s}} \|u\|_{L_t^\infty L_x^{\frac{2r}{d}}}^{1-\frac{1}{s}}.$$

Thus, the claim will follow once we establish

$$\big\||\nabla|^{s-1} O\big(|u|^{\frac{4}{d}}\big)\big\|_{L_t^\infty L_x^{\frac{2rs}{(r-d)(s-1)+4s}}} \lesssim \big\||\nabla|^s u\big\|_{L_t^\infty L_x^2}^{1-\frac{1}{s}} \|u\|_{L_t^\infty L_x^{\frac{2r}{d}}}^{\frac{4}{d}+\frac{1}{s}-1}. \tag{22}$$

Applying Lemma 2.4, we obtain

$$\big\||\nabla|^{s-1} O\big(|u|^{\frac{4}{d}}\big)\big\|_{L_t^\infty L_x^{\frac{2rs}{(r-d)(s-1)+4s}}} \lesssim \big\||\nabla|^\sigma u\big\|_{L_t^\infty L_x^{\frac{2rs}{sd+\sigma(r-d)}}}^{\frac{s-1}{\sigma}} \|u\|_{L_t^\infty L_x^{\frac{2r}{d}}}^{\frac{4}{d}-\frac{s-1}{\sigma}}$$

for any $\frac{d(s-1)}{4} < \sigma < 1$. The inequality (22) now follows from

$$\big\||\nabla|^\sigma u\big\|_{L_t^\infty L_x^{\frac{2rs}{sd+\sigma(r-d)}}} \lesssim \big\||\nabla|^s u\big\|_{L_t^\infty L_x^2}^{\frac{\sigma}{s}} \|u\|_{L_t^\infty L_x^{\frac{2r}{d}}}^{1-\frac{\sigma}{s}},$$

which is a consequence of interpolation.

Note that the restriction $r \geq d$ guarantees that certain Lebesgue exponents appearing above lie in the range $[1, \infty]$. In fact, one may relax this restriction a little, but we will not need this here. $\square$

2.3. **Strichartz estimates.** Naturally, everything that we do for the nonlinear Schrödinger equation builds on basic properties of the linear propagator $e^{it\Delta}$.

From the explicit formula

$$e^{it\Delta} f(x) = \frac{1}{(4\pi i t)^{d/2}} \int_{\mathbb{R}^d} e^{i|x-y|^2/4t} f(y)\, dy,$$

we deduce the standard dispersive inequality

$$\|e^{it\Delta} f\|_{L^\infty(\mathbb{R}^d)} \lesssim \frac{1}{|t|^{d/2}} \|f\|_{L^1(\mathbb{R}^d)} \tag{23}$$

for all $t \neq 0$. Interpolating between this and the conservation of mass, gives

$$\|e^{it\Delta} f\|_{L^p(\mathbb{R}^d)} \lesssim |t|^{\frac{d}{p}-\frac{d}{2}} \|f\|_{L^{p'}(\mathbb{R}^d)} \tag{24}$$

for all $t \neq 0$ and $2 \leq p \leq \infty$. Here p' is the dual of p, that is, $\frac{1}{p} + \frac{1}{p'} = 1$.

Finer bounds on the (frequency localized) linear propagator can be derived using stationary phase:

Lemma 2.6 (Kernel estimates)**.** *For any $m \geq 0$, the kernel of the linear propagator obeys the following estimates:*

$$\big|(P_N e^{it\Delta})(x,y)\big| \lesssim_m \begin{cases} |t|^{-d/2} & : \ |x-y| \sim N|t| \\ \dfrac{N^d}{|N^2 t|^m \langle N|x-y|\rangle^m} & : \ \textit{otherwise} \end{cases} \tag{25}$$

for $|t| \geq N^{-2}$ and

$$\big|(P_N e^{it\Delta})(x,y)\big| \lesssim_m N^d \big\langle N|x-y|\big\rangle^{-m} \tag{26}$$

for $|t| \leq N^{-2}$.

We also record the following standard Strichartz estimates:

Lemma 2.7 (Strichartz)**.** *Let I be an interval, let $t_0 \in I$, and let $u_0 \in L^2_x(\mathbb{R}^d)$ and $f \in L^{2(d+2)/(d+4)}_{t,x}(I \times \mathbb{R}^d)$, with $d \geq 3$. Then, the function u defined by*

$$u(t) := e^{i(t-t_0)\Delta} u_0 - i \int_{t_0}^{t} e^{i(t-t')\Delta} f(t')\, dt'$$

obeys the estimate

$$\|u\|_{C^0_t L^2_x} + \|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}} + \|u\|_{L^2_t L_x^{\frac{2d}{d-2}}} \lesssim \|u_0\|_{L^2_x} + \|f\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}},$$

where all spacetime norms are over $I \times \mathbb{R}^d$.

Proof. See, for example, [19, 36]. For the endpoint see [22]. □

We will also need three variants of the Strichartz inequality. First, we observe a weighted Strichartz estimate, which exploits the spherical symmetry heavily in order to obtain spatial decay. It is very useful in regions of space far from the origin $x = 0$.

Lemma 2.8 (Weighted Strichartz)**.** *Let I be an interval, let $t_0 \in I$, and let $u_0 \in L^2_x(\mathbb{R}^d)$ and $f \in L^{2(d+2)/(d+4)}_{t,x}(I \times \mathbb{R}^d)$ be spherically symmetric. Then, the function u defined by*

$$u(t) := e^{i(t-t_0)\Delta} u_0 - i \int_{t_0}^{t} e^{i(t-t')\Delta} f(t')\, dt'$$

obeys the estimate

$$\big\| |x|^{\frac{2(d-1)}{q}} u \big\|_{L^q_t L_x^{\frac{2q}{q-4}}(I \times \mathbb{R}^d)} \lesssim \|u_0\|_{L^2_x(\mathbb{R}^d)} + \|f\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}(I \times \mathbb{R}^d)}$$

for all $4 \leq q \leq \infty$.

Proof. For $q = \infty$, this corresponds to the trivial endpoint in Strichartz inequality. We will only prove the result for the $q = 4$ endpoint, since the remaining cases then follow by interpolation.

As in the usual proof of Strichartz inequality, the method of TT^* together with the Christ–Kiselev lemma and Hardy–Littlewood–Sobolev inequality reduce matters to proving that

$$\big\| |x|^{\frac{(d-1)}{2}} e^{it\Delta} |x|^{\frac{(d-1)}{2}} g \big\|_{L_x^\infty(\mathbb{R}^d)} \lesssim |t|^{-\frac{1}{2}} \|g\|_{L_x^1(\mathbb{R}^d)} \tag{27}$$

for all radial functions g.

Let P_{rad} denote the projection onto radial functions. Then

$$[e^{it\Delta} P_{\text{rad}}](x,y) = (4\pi i t)^{-\frac{d}{2}} e^{i\frac{|x|^2+|y|^2}{4t}} \int_{S^{d-1}} e^{-i\frac{|y|\omega\cdot x}{2t}}\, d\sigma(\omega),$$

where $d\sigma$ denotes the uniform probability measure on the unit sphere S^{d-1}. This integral can be evaluated exactly in terms the $J_{\frac{d-2}{2}}$ Bessel function. Using this, or simple stationary phase arguments, one sees that

$$\big|[e^{it\Delta} P_{\text{rad}}](x,y)\big| \lesssim |t|^{-\frac{d}{2}} \big(\tfrac{|y||x|}{|t|}\big)^{-\frac{d-1}{2}} \lesssim |t|^{-\frac{1}{2}} |x|^{-\frac{d-1}{2}} |y|^{-\frac{d-1}{2}}.$$

The radial dispersive estimate (27) now follows easily. $\square$

We will rely crucially on a slightly different type of improvement to the Strichartz inequality in the spherically symmetric case due to Shao [33], which improves the spacetime decay of the solution after localizing in frequency:

Lemma 2.9 (Shao's Strichartz Estimate, [33, Corollary 6.2])**.** *For $f \in L^2_{rad}(\mathbb{R}^d)$ we have*

$$\|P_N e^{it\Delta} f\|_{L^q_{t,x}(\mathbb{R}\times\mathbb{R}^d)} \lesssim_q N^{\frac{d}{2}-\frac{d+2}{q}} \|f\|_{L_x^2(\mathbb{R}^d)}, \tag{28}$$

provided $q > \frac{4d+2}{2d-1}$.

The last result is a bilinear estimate, which will be useful for controlling interactions between widely separated frequencies.

Lemma 2.10 (Bilinear Strichartz)**.** *For any spacetime slab $I \times \mathbb{R}^d$, any $t_0 \in I$, and any $M, N > 0$, we have*

$$\begin{aligned} &\|(P_{\geq N} u)(P_{\leq M} v)\|_{L^2_{t,x}(I\times\mathbb{R}^d)} \\ &\qquad \lesssim_q N^{-\frac{1}{2}} M^{\frac{d-1}{2}} \Big(\|P_{\geq N} u(t_0)\|_{L^2} + \|(i\partial_t + \Delta) P_{\geq N} u\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}(I\times\mathbb{R}^d)} \Big) \\ &\qquad\qquad \times \Big(\|P_{\leq M} v(t_0)\|_{L^2} + \|(i\partial_t + \Delta) P_{\leq M} v\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}(I\times\mathbb{R}^d)} \Big), \end{aligned}$$

for all spherically symmetric functions u, v on I.

Proof. See [44, Lemma 2.5], which builds on earlier versions in [6, 14]. $\square$

3. The self-similar solution

In this section we preclude self-similar solutions. As mentioned in the Introduction, the key ingredient is additional regularity.

Theorem 3.1 (Regularity in the self-similar case)**.** *Let $d \geq 3$ and let u be a spherically symmetric solution to* (1) *that is almost periodic modulo scaling and that is self-similar in the sense of Theorem 1.10. Then $u(t) \in H^s_x(\mathbb{R}^d)$ for all $t \in (0,\infty)$ and all $0 \leq s < 1 + \frac{4}{d}$.*

Corollary 3.2 (Absence of self-similar solutions)**.** *For $d \geq 3$ there are no non-zero spherically symmetric solutions to* (1) *that are self-similar in the sense of Theorem 1.10.*

Proof. By Theorem 3.1, any such solution would obey $u(t) \in H^1_x(\mathbb{R}^d)$ for all $t \in (0,\infty)$. Then, by the H^1_x global well-posedness theory described after Theorem 1.5, there exists a global solution with initial data $u(t_0)$ at any time $t_0 \in (0,\infty)$; recall that we assume $M(u) < M(Q)$ in the focusing case. On the other hand, self-similar solutions blow up at time $t = 0$. These two facts (combined with the uniqueness statement in Theorem 1.3) yield a contradiction. □

The remainder of this section is devoted to proving Theorem 3.1.

Let u be as in Theorem 3.1. For any $A > 0$, we define

$$\begin{aligned}
\mathcal{M}(A) &:= \sup_{T>0} \|u_{>AT^{-1/2}}(T)\|_{L^2_x(\mathbb{R}^d)} \\
\mathcal{S}(A) &:= \sup_{T>0} \|u_{>AT^{-1/2}}\|_{L^{2(d+2)/d}_{t,x}([T,2T]\times\mathbb{R}^d)} \\
\mathcal{N}(A) &:= \sup_{T>0} \|P_{>AT^{-1/2}} F(u)\|_{L^{2(d+2)/(d+4)}_{t,x}([T,2T]\times\mathbb{R}^d)}.
\end{aligned} \tag{29}$$

The notation chosen indicates the quantity being measured, namely, the mass, the symmetric Strichartz norm, and the nonlinearity in the adjoint Strichartz norm, respectively. As u is self-similar, $N(t)$ is comparable to $T^{-1/2}$ for t in the interval $[T, 2T]$. Thus, the Littlewood-Paley projections are adapted to the natural frequency scale on each dyadic time interval.

To prove Theorem 3.1 it suffices to show that for every $0 < s < 1 + \frac{4}{d}$ we have

$$\mathcal{M}(A) \lesssim_{s,u} A^{-s}, \tag{30}$$

whenever A is sufficiently large depending on u and s. To establish this, we need a variety of estimates linking $\mathcal{M}$, $\mathcal{S}$, and $\mathcal{N}$. From mass conservation, Lemma 1.12, self-similarity, and Hölder's inequality, we see that

$$\mathcal{M}(A) + \mathcal{S}(A) + \mathcal{N}(A) \lesssim_u 1 \tag{31}$$

for all $A > 0$. From the Strichartz inequality (Lemma 2.7), we also see that

$$\mathcal{S}(A) \lesssim \mathcal{M}(A) + \mathcal{N}(A) \tag{32}$$

for all $A > 0$. Another application of Strichartz shows

$$\|u\|_{L^2_t L^{\frac{2d}{d-2}}_x([T,2T]\times\mathbb{R}^d)} \lesssim_u 1. \tag{33}$$

Next, we obtain a deeper connection between these quantities.

Lemma 3.3 (Nonlinear estimate)**.** *Let $\eta > 0$ and $0 < s < 1 + \frac{4}{d}$. For all $A > 100$ and $0 < \beta \leq 1$, we have*

$$\begin{aligned}
\mathcal{N}(A) \lesssim_u \sum_{N \leq \eta A^\beta} \bigl(\tfrac{N}{A}\bigr)^s \mathcal{S}(N) &+ \bigl[\mathcal{S}(\eta A^{\frac{\beta}{2(d-1)}}) + \mathcal{S}(\eta A^\beta)\bigr]^{\frac{4}{d}} \mathcal{S}(\eta A^\beta) \\
&+ A^{-\frac{2\beta}{d^2}} \bigl[\mathcal{M}(\eta A^\beta) + \mathcal{N}(\eta A^\beta)\bigr].
\end{aligned} \tag{34}$$

Proof. Fix $\eta > 0$ and $0 < s < 1 + \frac{4}{d}$. It suffices to bound

$$\|P_{>AT^{-\frac{1}{2}}} F(u)\|_{L^{2(d+2)/(d+4)}_{t,x}([T,2T]\times\mathbb{R}^d)}$$

by the right-hand side of (34) for arbitrary $T > 0$ and all $A > 100$ and $0 < \beta \leq 1$.

To achieve this, we decompose

$$\begin{aligned} F(u) = F(u_{\leq \eta A^\beta T^{-\frac{1}{2}}}) + O\big(|u_{\leq \eta A^\alpha T^{-\frac{1}{2}}}|^{\frac{4}{d}} |u_{>\eta A^\beta T^{-\frac{1}{2}}}|\big) \\ + O\big(|u_{\eta A^\alpha T^{-\frac{1}{2}} < \cdot \leq \eta A^\beta T^{-\frac{1}{2}}}|^{\frac{4}{d}} |u_{>\eta A^\beta T^{-\frac{1}{2}}}|\big) + O\big(|u_{>\eta A^\beta T^{-\frac{1}{2}}}|^{1+\frac{4}{d}}\big), \end{aligned} \tag{35}$$

where $\alpha = \frac{\beta}{2(d-1)}$. To estimate the contribution from the last two terms in the expansion above, we discard the projection to high frequencies and then use Hölder's inequality and (29):

$$\begin{aligned} \big\| |u_{\eta A^\alpha T^{-\frac{1}{2}} < \cdot \leq \eta A^\beta T^{-\frac{1}{2}}}|^{\frac{4}{d}} u_{>\eta A^\beta T^{-\frac{1}{2}}} \big\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}([T,2T]\times\mathbb{R}^d)} &\lesssim \mathcal{S}(\eta A^\alpha)^{\frac{4}{d}} \mathcal{S}(\eta A^\beta) \\ \big\| |u_{>\eta A^\beta T^{-\frac{1}{2}}}|^{1+\frac{4}{d}} \big\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}([T,2T]\times\mathbb{R}^d)} &\lesssim \mathcal{S}(\eta A^\beta)^{1+\frac{4}{d}}. \end{aligned}$$

To estimate the contribution coming from second term on the right-hand side of (35), we discard the projection to high frequencies and then use Hölder's inequality, Lemma 2.2, Lemma 2.10, and (32):

$$\begin{aligned} &\big\|P_{>AT^{-\frac{1}{2}}} O\big(|u_{\leq \eta A^\alpha T^{-\frac{1}{2}}}|^{\frac{4}{d}} |u_{>\eta A^\beta T^{-\frac{1}{2}}}|\big)\big\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}([T,2T]\times\mathbb{R}^d)} \\ &\lesssim \big\|u_{\leq \eta A^\alpha T^{-\frac{1}{2}}} u_{>\eta A^\beta T^{-\frac{1}{2}}}\big\|^{\frac{8}{d^2}}_{L^2_{t,x}([T,2T]\times\mathbb{R}^d)} \big\|u_{>\eta A^\beta T^{-\frac{1}{2}}}\big\|^{1-\frac{8}{d^2}}_{L_{t,x}^{\frac{2(d+2)}{d}}([T,2T]\times\mathbb{R}^d)} \\ &\qquad \times \big\|u_{\leq \eta A^\alpha T^{-\frac{1}{2}}}\big\|^{\frac{4}{d}-\frac{8}{d^2}}_{L^2_{t,x}([T,2T]\times\mathbb{R}^d)} \\ &\lesssim_u \big[(\eta A^\beta T^{-\frac{1}{2}})^{-\frac{1}{2}} (\eta A^\alpha T^{-\frac{1}{2}})^{\frac{d-1}{2}}\big]^{\frac{8}{d^2}} \big[\mathcal{M}(\eta A^\beta) + \mathcal{N}(\eta A^\beta)\big]^{\frac{8}{d^2}} \mathcal{S}(\eta A^\beta)^{1-\frac{8}{d^2}} T^{\frac{2}{d}-\frac{4}{d^2}} \\ &\lesssim_u A^{-\frac{2\beta}{d^2}} \big[\mathcal{M}(\eta A^\beta) + \mathcal{N}(\eta A^\beta)\big]. \end{aligned}$$

We now turn to the first term on the right-hand side of (35). By Lemma 2.2 and Corollary 2.5 combined with (31), we estimate

$$\begin{aligned} \|P_{>AT^{-\frac{1}{2}}} F(u_{\leq \eta A^\beta T^{-\frac{1}{2}}})\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}([T,2T]\times\mathbb{R}^d)} &\lesssim (AT^{-\frac{1}{2}})^{-s} \big\||\nabla|^s F(u_{\leq \eta A^\beta T^{-\frac{1}{2}}})\big\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}([T,2T]\times\mathbb{R}^d)} \\ &\lesssim_u (AT^{-\frac{1}{2}})^{-s} \big\||\nabla|^s u_{\leq \eta A^\beta T^{-\frac{1}{2}}}\big\|_{L_{t,x}^{\frac{2(d+2)}{d}}([T,2T]\times\mathbb{R}^d)} \\ &\lesssim_u \sum_{N \leq \eta A^\beta} \big(\tfrac{N}{A}\big)^s \mathcal{S}(N), \end{aligned}$$

which is acceptable. This finishes the proof of the lemma. □

We have some decay as $A \to \infty$:

Lemma 3.4 (Qualitative decay)**.** *We have*

$$\lim_{A\to\infty} \mathcal{M}(A) = \lim_{A\to\infty} \mathcal{S}(A) = \lim_{A\to\infty} \mathcal{N}(A) = 0. \tag{36}$$

Proof. The vanishing of the first limit follows from Definition 1.8, self-similarity, and (29). By interpolation, (29), and (33),

$$\mathcal{S}(A) \lesssim \mathcal{M}(A)^{\frac{2}{d+2}} \|u_{\geq AT^{-\frac{1}{2}}}\|_{L_t^2 L_x^{\frac{2d}{d-2}}([T,2T]\times\mathbb{R}^d)}^{\frac{d}{d+2}} \lesssim_u \mathcal{M}(A)^{\frac{2}{d+2}}.$$

Thus, as the first limit in (36) vanishes, we obtain that the second limit vanishes. The vanishing of the third limit follows from that of the second and Lemma 3.3. □

We have now gathered enough tools to prove some regularity, albeit in the symmetric Strichartz space. As such, the next result is the crux of this section.

Proposition 3.5 (Quantitative decay estimate)**.** *Let $0<\eta<1$ and $0<s<1+\frac{4}{d}$. If η is sufficiently small depending on u and s, and A is sufficiently large depending on u, s, and η,*

$$\mathcal{S}(A) \leq \sum_{N\leq \eta A} \left(\tfrac{N}{A}\right)^s \mathcal{S}(N) + A^{-\frac{1}{d^2}}. \tag{37}$$

In particular,

$$\mathcal{S}(A) \lesssim_u A^{-\frac{1}{d^2}}, \tag{38}$$

for all $A>0$.

Proof. Fix $\eta\in(0,1)$ and $0<s<1+\frac{4}{d}$. To establish (37), it suffices to show

$$\|u_{>AT^{-1/2}}\|_{L_{t,x}^{\frac{2(d+2)}{d}}([T,2T]\times\mathbb{R}^d)} \lesssim_u \sum_{N\leq\eta A} \left(\tfrac{N}{A}\right)^{s+\varepsilon} \mathcal{S}(N) + A^{-\frac{3}{2d^2}} \tag{39}$$

for all $T>0$ and some small $\varepsilon>0$, since then (37) follows by requiring η to be small and A to be large, both depending upon u.

Fix $T>0$. By writing the Duhamel formula (4) beginning at $\frac{T}{2}$ and then using Lemma 2.7, we obtain

$$\begin{aligned}\|u_{>AT^{-1/2}}\|_{L_{t,x}^{\frac{2(d+2)}{d}}([T,2T]\times\mathbb{R}^d)} &\lesssim \|P_{>AT^{-1/2}} e^{i(t-\frac{T}{2})\Delta} u(\tfrac{T}{2})\|_{L_{t,x}^{\frac{2(d+2)}{d}}([T,2T]\times\mathbb{R}^d)} \\ &\quad + \|P_{>AT^{-1/2}} F(u)\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}([\frac{T}{2},2T]\times\mathbb{R}^d)}.\end{aligned}$$

First, we consider the second term. By (29), we have

$$\|P_{>AT^{-1/2}} F(u)\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}([\frac{T}{2},2T]\times\mathbb{R}^d)} \lesssim \mathcal{N}(A/2).$$

Using Lemma 3.3 (with $\beta=1$ and s replaced by $s+\varepsilon$ for some $0<\varepsilon<1+\frac{4}{d}-s$) combined with Lemma 3.4 (choosing A sufficiently large depending on u, s, and η), and (31), we derive

$$\|P_{>AT^{-1/2}} F(u)\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}([\frac{T}{2},2T]\times\mathbb{R}^d)} \lesssim_u \mathrm{RHS}(39).$$

Thus, the second term is acceptable.

We now consider the first term. It suffices to show

$$\|P_{>AT^{-1/2}} e^{i(t-\frac{T}{2})\Delta} u(\tfrac{T}{2})\|_{L_{t,x}^{\frac{2(d+2)}{d}}([T,2T]\times\mathbb{R}^d)} \lesssim_u A^{-\frac{3}{2d^2}}, \tag{40}$$

which we will deduce by first proving two estimates at a single frequency scale, interpolating between them, and then summing.

From Lemma 2.9 and mass conservation, we have

$$\|P_{BT^{-1/2}} e^{i(t-\frac{T}{2})\Delta} u(\tfrac{T}{2})\|_{L^q_{t,x}([T,2T]\times\mathbb{R}^d)} \lesssim_{u,q} (BT^{-1/2})^{\frac{d}{2}-\frac{d+2}{q}} \tag{41}$$

for all $\frac{4d+2}{2d-1} < q \leq \frac{2(d+2)}{d}$ and $B > 0$. This is our first estimate.

Using the Duhamel formula (4), we write

$$P_{BT^{-1/2}} e^{i(t-\frac{T}{2})\Delta} u(\tfrac{T}{2}) = P_{BT^{-1/2}} e^{i(t-\varepsilon)\Delta} u(\varepsilon) - i\int_\varepsilon^{\frac{T}{2}} P_{BT^{-1/2}} e^{i(t-t')\Delta} F(u(t'))\, dt'$$

for any $\varepsilon > 0$. By self-similarity, the former term converges strongly to zero in L^2_x as $\varepsilon \to 0$. Convergence to zero in $L_x^{2d/(d-2)}$ then follows from Lemma 2.2. Thus, using Hölder's inequality followed by the dispersive estimate (24), and then (33), we estimate

$$\begin{aligned}
\|P_{BT^{-1/2}} &e^{i(t-\frac{T}{2})\Delta} u(\tfrac{T}{2})\|_{L_{t,x}^{\frac{2d}{d-2}}([T,2T]\times\mathbb{R}^d)} \\
&\lesssim T^{\frac{d-2}{2d}} \Big\| \int_0^{\frac{T}{2}} \frac{1}{t-t'} \|F(u(t'))\|_{L_x^{\frac{2d}{d+2}}}\, dt' \Big\|_{L_t^\infty([T,2T])} \\
&\lesssim T^{-\frac{d+2}{2d}} \|F(u)\|_{L^1_t L_x^{\frac{2d}{d+2}}((0,\frac{T}{2}]\times\mathbb{R}^d)} \\
&\lesssim T^{-\frac{d+2}{2d}} \sum_{0<\tau\leq\frac{T}{4}} \|F(u)\|_{L^1_t L_x^{\frac{2d}{d+2}}([\tau,2\tau]\times\mathbb{R}^d)} \\
&\lesssim T^{-\frac{d+2}{2d}} \sum_{0<\tau\leq\frac{T}{4}} \tau^{1/2} \|u\|_{L^2_t L_x^{\frac{2d}{d-2}}([\tau,2\tau]\times\mathbb{R}^d)} \|u\|^{\frac{4}{d}}_{L^\infty_t L^2_x([\tau,2\tau]\times\mathbb{R}^d)} \\
&\lesssim_u T^{-1/d}.
\end{aligned}$$

Interpolating between the estimate just proved and (41) with $q = \frac{2d(d+2)(4d-3)}{4d^3-3d^2+12}$, we obtain

$$\|P_{BT^{-1/2}} e^{i(t-\frac{T}{2})\Delta} u(\tfrac{T}{2})\|_{L_{t,x}^{\frac{2(d+2)}{d}}([T,2T]\times\mathbb{R}^d)} \lesssim_u B^{-\frac{3}{2d^2}}.$$

Summing this over dyadic $B \geq A$ yields (40) and hence (39).

We now justify (38). Given an integer $K \geq 4$, we set $\eta = 2^{-K}$. Then, there exists A_0 depending on u and K, so that (37) holds for $A \geq A_0$. By (31), we need only bound $\mathcal{S}(A)$ for $A \geq A_0$.

Let $k \geq 0$ and set $A = 2^k A_0$ in (37). Then, writing $N = 2^l A_0$ and using (31),

$$\begin{aligned}
\mathcal{S}(2^k A_0) &\leq \sum_{l\leq k-K} 2^{-(k-l)s} \mathcal{S}(2^l A_0) + (2^k A_0)^{-\beta} \\
&\leq \sum_{l=0}^{k-K} 2^{-(k-l)s} \mathcal{S}(2^l A_0) + \frac{2^{-ks}}{1-2^{-s}} \mathcal{S}(0) + 2^{-k\beta} A_0^{-\beta},
\end{aligned}$$

where $\beta := d^{-2}$. Setting $s = 1$ and applying Lemma 2.1 with $x_k = \mathcal{S}(2^k A_0)$ and $b_k = O_u(2^{-k\beta})$, we deduce

$$\mathcal{S}(2^k A_0) \lesssim_u 2^{-k/d^2},$$

provided K is chosen sufficiently large. This gives the necessary bound on $\mathcal{S}$. $\square$

Corollary 3.6. *For any $A > 0$ we have*

$$\mathcal{M}(A) + \mathcal{S}(A) + \mathcal{N}(A) \lesssim_u A^{-1/d^2}.$$

Proof. The bound on $\mathcal{S}$ was proved in the previous proposition. The bound on $\mathcal{N}$ follows from this, Lemma 3.3 with $\beta = 1$, and (31).

We now turn to the bound on $\mathcal{M}$. By Lemma 1.11,

$$\|P_{>AT^{-1/2}}u(T)\|_2 \lesssim \sum_{k=0}^{\infty} \Big\| \int_{2^kT}^{2^{k+1}T} e^{i(T-t')\Delta} P_{>AT^{-1/2}} F(u(t'))\, dt' \Big\|_2, \tag{42}$$

where weak convergence has become strong convergence because of the frequency projection and the fact that $N(t) = t^{-1/2} \to 0$ as $t \to \infty$. Intuitively, the reason for using (15) forward in time is that the solution becomes smoother as $N(t) \to 0$.

Combining (42) with Lemma 2.7 and (29), we get

$$\mathcal{M}(A) = \sup_{T>0} \|P_{>AT^{-1/2}}u(T)\|_2 \lesssim \sum_{k=0}^{\infty} \mathcal{N}(2^{k/2}A). \tag{43}$$

The desired bound on $\mathcal{M}$ now follows from that on $\mathcal{N}$. $\square$

Proof of Theorem 3.1. Let $0 < s < 1+\frac{4}{d}$. Combining Lemma 3.3 (with $\beta = 1-\frac{1}{2d^2}$), (32), and (43), we deduce that if

$$\mathcal{S}(A) + \mathcal{M}(A) + \mathcal{N}(A) \lesssim_u A^{-\sigma}$$

for some $0 < \sigma < s$, then

$$\mathcal{S}(A) + \mathcal{M}(A) + \mathcal{N}(A) \lesssim_u A^{-\sigma}\Big(A^{-\frac{s-\sigma}{2d^2}} + A^{-\frac{(d+1)(3d-2)\sigma}{2d^3(d-1)}} + A^{-\frac{3-\sigma}{2d^2}-\frac{d^2-2}{2d^4}}\Big).$$

More precisely, Lemma 3.3 provides the bound on $\mathcal{N}(A)$, then (43) gives the bound on $\mathcal{M}(A)$ and then finally (32) gives the bound on $\mathcal{S}(A)$.

Iterating this statement shows that $u(t) \in H^s_x(\mathbb{R}^d)$ for all $0 < s < 1 + \frac{4}{d}$. Note that Corollary 3.6 allows us to begin the iteration with $\sigma = d^{-2}$. $\square$

4. An in/out decomposition

In this section, we will often write radial functions on $\mathbb{R}^d$ just in terms of the radial variable. With this convention,

$$f(r) = r^{\frac{2-d}{2}} \int_0^\infty J_{\frac{d-2}{2}}(kr)\hat{f}(k)\, k^{\frac{d}{2}}\, dk \quad \text{and} \quad \hat{f}(k) = k^{\frac{2-d}{2}} \int_0^\infty J_{\frac{d-2}{2}}(kr) f(r)\, r^{\frac{d}{2}}\, dr,$$

as can be seen from [35, Theorem IV.3.3]. Here J_ν denotes the Bessel function of order ν. In particular, $g(k,r) := r^{\frac{2-d}{2}} J_{\frac{d-2}{2}}(kr)$ solves the radial Helmholtz equation

$$- g_{rr} - \tfrac{d-1}{r} g_r = k^2 g, \tag{44}$$

which corresponds to the fact that $g(k,r)$ represents a spherical standing wave of frequency $k^2/(2\pi)$. Incoming and outgoing spherical waves are represented by two further solutions of (44), namely,

$$g_-(k,r) := r^{\frac{2-d}{2}} H^{(2)}_{\frac{d-2}{2}}(kr) \quad \text{and} \quad g_+(k,r) := r^{\frac{2-d}{2}} H^{(1)}_{\frac{d-2}{2}}(kr),$$

respectively. Note that $g = \frac{1}{2}g_+ + \frac{1}{2}g_-$. This leads us to define the projection onto outgoing spherical waves by

$$\begin{aligned}
[P^+ f](r) &= \tfrac{1}{2}\int_0^\infty r^{\frac{2-d}{2}} H^{(1)}_{\frac{d-2}{2}}(kr)\hat{f}(k)\, k^{\frac{d}{2}}\, dk \\
&= \tfrac{1}{2} r^{\frac{2-d}{2}} \int_0^\infty \Big[\int_0^\infty H^{(1)}_{\frac{d-2}{2}}(kr) J_{\frac{d-2}{2}}(k\rho)\, k\, dk\Big] f(\rho)\, \rho^{\frac{d}{2}}\, d\rho \\
&= \tfrac{1}{2} f(r) + \tfrac{i}{\pi}\int_0^\infty \frac{r^{2-d}\, f(\rho)\,\rho^{d-1}\, d\rho}{r^2-\rho^2}.
\end{aligned} \tag{45}$$

In order to derive the last equality we used [20, §6.521.2] together with analytic continuation. Similarly, we define the projection onto incoming waves by

$$\begin{aligned}
[P^- f](r) &= \tfrac{1}{2}\int_0^\infty r^{\frac{2-d}{2}} H^{(2)}_{\frac{d-2}{2}}(kr)\hat{f}(k)\, k^{\frac{d}{2}}\, dk \\
&= \tfrac{1}{2} f(r) - \tfrac{i}{\pi}\int_0^\infty \frac{r^{2-d}\, f(\rho)\,\rho^{d-1}\, d\rho}{r^2-\rho^2}.
\end{aligned}$$

Note that the kernel of P^- is the complex conjugate of that belonging to P^+, as is required by time-reversal symmetry.

We will write $P^\pm_N$ for the product $P^\pm P_N$.

Remark. For $f(\rho) \in L^2(\rho^{d-1}\, d\rho)$,

$$\int_0^\infty \big|f(\rho)\big|^2 \rho^{d-1}\, d\rho = \tfrac{1}{2}\int \big|s^{\frac{d-2}{4}} f(\sqrt{s})\big|^2\, ds$$

and with $t = r^2$,

$$\int_0^\infty \frac{r^{2-d}\, f(\rho)\,\rho^{d-1}\, d\rho}{r^2-\rho^2} = \tfrac{1}{2} t^{-\frac{d-2}{4}} \int_0^\infty \Big(\frac{s}{t}\Big)^{\frac{d-2}{4}} \frac{s^{\frac{d-2}{4}} f(\sqrt{s})\, ds}{t-s}. \tag{46}$$

Thus $P^+ : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is bounded if and only if the Hilbert transform is bounded in the weighted space $L^2([0,\infty), t^{-(d-2)/2}\, dt)$. Thus P^+ is unbounded on $L^2(\mathbb{R}^d)$ for $d \geq 4$.

Lemma 4.1 (Kernel estimates)**.** *For $|x| \gtrsim N^{-1}$ and $t \gtrsim N^{-2}$, the integral kernel obeys*

$$\big|[P^\pm_N e^{\mp it\Delta}](x,y)\big| \lesssim \begin{cases} \big(|x||y|\big)^{-\frac{d-1}{2}} |t|^{-\frac{1}{2}} & : \ |y|-|x| \sim Nt \\ \frac{N^d}{(N|x|)^{\frac{d-1}{2}} \langle N|y|\rangle^{\frac{d-1}{2}}} \big\langle N^2 t + N|x| - N|y|\big\rangle^{-m} & : \ \textit{otherwise} \end{cases}$$

for any $m \geq 0$. For $|x| \gtrsim N^{-1}$ and $|t| \lesssim N^{-2}$, the integral kernel obeys

$$\big|[P^\pm_N e^{\mp it\Delta}](x,y)\big| \lesssim \frac{N^d}{(N|x|)^{\frac{d-1}{2}} \langle N|y|\rangle^{\frac{d-1}{2}}} \big\langle N|x| - N|y|\big\rangle^{-m}$$

for any $m \geq 0$.

Proof. The proof is an exercise in stationary phase. We will only provide the details for $P^+_N e^{-it\Delta}$, the other kernel being its complex conjugate. By (45) we have the following formula for the kernel:

$$[P^+_N e^{-it\Delta}](x,y) = \tfrac{1}{2}\big(|x||y|\big)^{-\frac{d-2}{2}} \int_0^\infty H^{(1)}_{\frac{d-2}{2}}(k|x|) J_{\frac{d-2}{2}}(k|y|) e^{itk^2} \psi\big(\tfrac{k}{N}\big)\, k\, dk \tag{47}$$

where ψ is the multiplier from the Littlewood–Paley projection. To proceed, we use the following information about Bessel/Hankel functions:

$$J_{\frac{d-2}{2}}(r) = \frac{a(r)e^{ir}}{\langle r\rangle^{1/2}} + \frac{\bar{a}(r)e^{-ir}}{\langle r\rangle^{1/2}}, \tag{48}$$

where $a(r)$ obeys the symbol estimates

$$\Big|\frac{\partial^m a(r)}{\partial r^m}\Big| \lesssim \langle r\rangle^{-m} \quad \text{for all } m \geq 0. \tag{49}$$

The Hankel function $H^{(1)}_{\frac{d-2}{2}}(r)$ has a singularity at $r = 0$; however, for $r \gtrsim 1$,

$$H^{(1)}_{\frac{d-2}{2}}(r) = \frac{b(r)e^{ir}}{r^{1/2}} \tag{50}$$

for a smooth function $b(r)$ obeying (49). As we assume $|x| \gtrsim N^{-1}$, the singularity does not enter into our considerations.

Substituting (48) and (50) into (47), we see that a stationary phase point can only occur in the term containing $\bar{a}(r)$ and even then only if $|y| - |x| \sim Nt$. In this case, stationary phase yields the first estimate. In all other cases, integration by parts yields the second estimate.

The short-time estimate is also a consequence of (47) and stationary phase techniques. Since t is so small, e^{ik^2t} shows no appreciable oscillation and can be incorporated into $\psi(\frac{k}{N})$. For $\big||y| - |x|\big| \leq N^{-1}$, the result follows from the naive L^1 estimate. For larger $|x| - |y|$, one integrates by parts m times. $\square$

Lemma 4.2 (Properties of $P^{\pm}$)**.**
(i) *$P^+ + P^-$ acts as the identity on $L^2_{rad}(\mathbb{R}^d)$.*
(ii) *Fix $N > 0$. For any spherically symmetric function $f \in L^2_x(\mathbb{R}^d)$,*

$$\big\|P^{\pm}P_{\geq N}f\big\|_{L^2_x(|x|\geq \frac{1}{100}N^{-1})} \lesssim \big\|f\big\|_{L^2_x(\mathbb{R}^d)}$$

with an N-independent constant.

Proof. Part (i) is immediate from the definition.

We turn now to part (ii). We only prove the inequality for P^+, as the result for P^- can be deduced from this. Let χ be a non-negative smooth function on $\mathbb{R}^+$ vanishing in a neighborhood of the origin and obeying $\chi(r) = 1$ for $r \geq \frac{1}{100}$. With this definition and (45),

$$\begin{aligned}\big\|P^{\pm}P_{\geq N}f\big\|^2_{L^2_x(|x|\geq N^{-1})} &\leq \big\|\chi(N|x|)P^{\pm}P_{\geq N}f\big\|^2_{L^2_x(\mathbb{R}^d)}\\ &= \int_0^\infty \Big|\int_0^\infty H^{(1)}_{\frac{d-2}{2}}(kr)\hat{f}(k)k^{\frac{d}{2}}\big(1-\phi\big(\tfrac{k}{N}\big)\big)\,dk\Big|^2 \chi(Nr)^2 r\,dr,\end{aligned}$$

where ϕ is the Littlewood-Paley cutoff, as in subsection 2.2. Note that by scaling, it suffices to treat the case $N = 1$. Because of the cutoffs, the only non-zero contribution comes from the region $kr \gtrsim 1$. This allows us to use the following information about Hankel functions: for $\rho \gtrsim 1$,

$$H^{(1)}_{\frac{d-2}{2}}(\rho) = \big(\tfrac{2}{\pi\rho}\big)^{\frac{1}{2}}[1 + b(\rho)]e^{i\rho - i(d-1)\frac{\pi}{4}}$$

where b is a symbol of order -1, that is,

$$\Big|\frac{\partial^m b(\rho)}{\partial \rho^m}\Big| \lesssim \langle\rho\rangle^{-m-1} \quad \text{for all } m \geq 0; \tag{51}$$

see for example [20]. Note that this is more refined than formula (50) used in the previous proof. With these observations, our goal has been reduced to showing that

$$\int_0^\infty \Big| \int_0^\infty e^{ikr}\big(1+b(kr)\big)\big(1-\phi(k)\big)g(k)\,dk\Big|^2 \chi(r)^2\,dr \lesssim \int_0^\infty |g(k)|^2\,dk$$

or, equivalently, that

$$K(k,k') := \big(1-\phi(k)\big)\big(1-\phi(k')\big)\int_0^\infty e^{i(k-k')r}\big(1+b(kr)\big)\big(1+\bar b(k'r)\big)\chi(r)^2\,dr$$

is the kernel of a bounded operator on $L^2_k([0,\infty))$. To this end, we will decompose K as the sum of two kernels, each of which we can estimate.

First, we consider

$$K_1(k,k') := \big(1-\phi(k)\big)\big(1-\phi(k')\big)\int_0^\infty e^{i(k-k')r}\chi(r)^2\,dr.$$

Without the prefactors, the integral is the kernel of a bounded Fourier multiplier and so a bounded operator on L^2_k. As ϕ is a bounded function, we may then deduce that K_1 is itself the kernel of a bounded operator.

Our second kernel is

$$K_2(k,k') := \big(1-\phi(k)\big)\big(1-\phi(k')\big)\int_0^\infty e^{i(k-k')r}\big[b(kr)+\bar b(k'r)+b(kr)\bar b(k'r)\big]\chi(r)^2\,dr,$$

which we will show is bounded using Schur's test. Note that the factors in front of the integral ensure that the kernel is zero unless $k\gtrsim 1$ and $k'\gtrsim 1$. By integration by parts, we see that

$$K_2(k,k') \lesssim_m |k-k'|^{-m}$$

for any $m\geq 1$, which offers ample control away from the diagonal. To obtain a good estimate near the diagonal, we need to break the integral into two pieces. We do this by writing $1=\chi(r/R)+(1-\chi(r/R))$, with $R\gg 1$. Integrating by parts once when r is large and not at all when r is small, leads to

$$\begin{aligned} K_2(k,k') &\lesssim \frac{1}{|k-k'|}\int \big[\tfrac{1}{kr^2}+\tfrac{1}{k'r^2}+\tfrac{1}{kk'r^3}\big]\chi\big(\tfrac{r}{R}\big)+\big[\tfrac{1}{kr}+\tfrac{1}{k'r}+\tfrac{1}{kk'r^2}\big]\tfrac{1}{R}\chi'\big(\tfrac{r}{R}\big)\,dr \\ &\qquad + \int \big[\tfrac{1}{kr}+\tfrac{1}{k'r}+\tfrac{1}{kk'r^2}\big]\chi(r)^2\big(1-\chi\big(\tfrac{r}{R}\big)\big)\,dr \\ &\lesssim \frac{1}{R|k-k'|}+\log(R). \end{aligned}$$

Choosing $R=|k-k'|^{-1}$ provides sufficient control near the diagonal to complete the application of Schur's test. □

5. Additional regularity

This section is devoted to a proof of

Theorem 5.1 (Regularity in the global case)**.** *Let $d\geq 3$ and let u be a global spherically symmetric solution to* (1) *that is almost periodic modulo scaling. Suppose also that $N(t)\lesssim 1$ for all $t\in\mathbb{R}$. Then $u\in L_t^\infty H_x^s(\mathbb{R}\times\mathbb{R}^d)$ for all $0\leq s<1+\frac{4}{d}$.*

The argument mimics that in [26], though the non-polynomial nature of the nonlinearity introduces several technical complications. That $u(t)$ is moderately smooth will follow from a careful study of the Duhamel formulae (15). Near t, we use the fact that there is little mass at high frequencies, as is implied by the

definition of almost periodicity and the boundedness of the frequency scale function $N(t)$. Far from t, we use the spherical symmetry of the solution. As this symmetry is only valuable at large radii, we are only able to exploit it by using the in/out decomposition described in Section 4.

Let us now begin the proof. For the remainder of the section, u will denote a solution to (1) that obeys the hypotheses of Theorem 5.1.

We first record some basic local estimates. From mass conservation we have

$$\|u\|_{L_t^\infty L_x^2(\mathbb{R}\times\mathbb{R}^d)} \lesssim_u 1, \tag{52}$$

while from Definition 1.8 and the fact that $N(t)$ is bounded we have

$$\lim_{N\to\infty} \|u_{\geq N}\|_{L_t^\infty L_x^2(\mathbb{R}\times\mathbb{R}^d)} = 0.$$

From Lemma 1.12 and $N(t) \lesssim 1$, we have

$$\|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}(J\times\mathbb{R}^d)} \lesssim_u \langle |J|\rangle^{\frac{d}{2(d+2)}} \tag{53}$$

for all intervals $J \subset \mathbb{R}$. By Hölder's inequality, this implies

$$\|F(u)\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}(J\times\mathbb{R}^d)} \lesssim_u \langle |J|\rangle^{\frac{d+4}{2(d+2)}} \tag{54}$$

and then, by the (endpoint) Strichartz inequality (Lemma 2.7),

$$\|u\|_{L_t^2 L_x^{\frac{2d}{d-2}}(J\times\mathbb{R}^d)} \lesssim_u \langle |J|\rangle^{\frac{1}{2}}. \tag{55}$$

More precisely, one first treats the case $|J| = O(1)$ using (53) and then larger intervals by subdivision. Similarly, from the weighted Strichartz inequality (Lemma 2.8),

$$\big\| |x|^{\frac{d-1}{2}} u_{N_1\leq\cdot\leq N_2}\big\|_{L_t^4 L_x^\infty(J\times\mathbb{R}^d)} \lesssim_u \langle |J|\rangle^{\frac{1}{4}} \tag{56}$$

uniformly in $0 < N_1 \leq N_2 < \infty$.

Now, for any dyadic number N, define

$$\mathcal{M}(N) := \|u_{\geq N}\|_{L_t^\infty L_x^2(\mathbb{R}\times\mathbb{R}^d)}. \tag{57}$$

From the discussion above, we see that $\mathcal{M}(N) \lesssim_u 1$ and

$$\lim_{N\to\infty} \mathcal{M}(N) = 0. \tag{58}$$

To prove Theorem 5.1, it suffices to show $\mathcal{M}(N) \lesssim_{u,s} N^{-s}$ for any $0 < s < 1 + \frac{4}{d}$ and all N sufficiently large depending on u and s. As we will explain momentarily, this will follow from Lemma 2.1 and the following

Proposition 5.2 (Regularity)**.** *Let u be as in Theorem 5.1, let $0 < s < 1 + \frac{4}{d}$, and let $\eta > 0$ be a small number. Then*

$$\mathcal{M}(N) \leq N^{-s} + \sum_{M\leq \eta N} \big(\tfrac{M}{N}\big)^s \mathcal{M}(M),$$

whenever N is sufficiently large depending on u, s, and η.

Indeed given $\varepsilon > 0$, let $\eta = 2^{-K}$ where K is so large that $2\log(K-1) < \varepsilon(K-1)$. Let N_0 be sufficiently large depending on u, s, and K so that the inequality in Proposition 5.2 holds for $N \geq N_0$. If we write $r = 2^{-s}$, $x_k = \mathcal{M}(2^k N_0)$, and

$$b_k = 2^{-ks} N_0^{-s} + \sum_{l\leq -1} 2^{-s(k-l)} \mathcal{M}(2^l N_0) \lesssim_u 2^{-ks} \lesssim_u 2^{-k(s-\varepsilon)},$$

then (16) holds. Therefore, $\mathcal{M}(N) \lesssim_{u,s} N^{\varepsilon-s}$ by the last sentence in Lemma 2.1.

The rest of this section is devoted to proving Proposition 5.2. Fix $0 < s < 1 + \frac{4}{d}$ and $\eta > 0$. Our task is to show that

$$\|u_{\geq N}(t_0)\|_{L^2_x(\mathbb{R}^d)} \leq N^{-s} + \sum_{M \leq \eta N} \big(\tfrac{M}{N}\big)^s \mathcal{M}(M)$$

for all times t_0 and all N sufficiently large (depending on u, s, and η). By time translation symmetry, we may assume $t_0 = 0$. As noted above, one of the keys to obtaining additional regularity is Lemma 1.11. Specifically, we have

$$\begin{aligned} u_{\geq N}(0) &= \big(P^+ + P^-\big) u_{\geq N}(0) \\ &= \lim_{T\to\infty} i \int_0^T P^+ e^{-it\Delta} P_{\geq N} F(u(t))\, dt - i \lim_{T\to\infty} \int_{-T}^0 P^- e^{-it\Delta} P_{\geq N} F(u(t))\, dt, \end{aligned} \tag{59}$$

where the limit is to be interpreted as a weak limit in L^2. However, this representation is not useful for $|x|$ small because the kernels of $P^\pm$ have a strong singularity at $x = 0$. To this end, we introduce the cutoff $\chi_N(x) := \chi(N|x|)$, where χ is the characteristic function of $[1, \infty)$. As short times and large times will be treated differently, we rewrite (59) as

$$\begin{aligned} &\chi_N(x) u_{\geq N}(0, x) \\ &\quad = i \int_0^\delta \chi_N(x) P^+ e^{-it\Delta} P_{\geq N} F(u(t))\, dt - i \int_{-\delta}^0 \chi_N(x) P^- e^{-it\Delta} P_{\geq N} F(u(t))\, dt \\ &\qquad + \lim_{T\to\infty} \sum_{M \geq N} i \int_\delta^T \int_{\mathbb{R}^d} \chi_N(x) [P^+_M e^{-it\Delta}](x, y) [\tilde{P}_M F(u(t))](y)\, dy\, dt \\ &\qquad - \lim_{T\to\infty} \sum_{M \geq N} i \int_{-T}^{-\delta} \int_{\mathbb{R}^d} \chi_N(x) [P^-_M e^{-it\Delta}](x, y) [\tilde{P}_M F(u(t))](y)\, dy\, dt, \end{aligned} \tag{60}$$

as weak limits in L^2_x. Note that we also used the identity

$$P_{\geq N} = \sum_{M \geq N} P_M \tilde{P}_M,$$

where $\tilde{P}_M := P_{M/2} + P_M + P_{2M}$, because of the way we will estimate the large-time integrals.

The analogous representation for treating small x is

$$\begin{aligned} &\big(1 - \chi_N(x)\big) u_{\geq N}(0, x) \\ &\quad = \lim_{T\to\infty} i \int_0^T \big(1 - \chi_N(x)\big) e^{-it\Delta} P_{\geq N} F(u(t))\, dt \\ &\quad = i \int_0^\delta \big(1 - \chi_N(x)\big) e^{-it\Delta} P_{\geq N} F(u(t))\, dt \\ &\qquad + \lim_{T\to\infty} \sum_{M \geq N} i \int_\delta^T \int_{\mathbb{R}^d} \big(1 - \chi_N(x)\big) [P_M e^{-it\Delta}](x, y) [\tilde{P}_M F(u(t))](y)\, dy\, dt, \end{aligned} \tag{61}$$

also as weak limits.

To deal with the poor nature of the limits in (60) and (61), we note that

$$f_T \to f \text{ weakly} \quad \Longrightarrow \quad \|f\| \leq \limsup_{T\to\infty} \|f_T\|, \tag{62}$$

or equivalently, that the unit ball is weakly closed.

Despite the fact that different representations will be used depending on the size of $|x|$, some estimates can be dealt with in a uniform manner. The first such example is a bound on integrals over short times.

Lemma 5.3 (Local estimate). *Let $0 < s < 1 + \frac{4}{d}$. For any sufficiently small $\eta > 0$, there exists $\delta = \delta(u, \eta) > 0$ such that*

$$\Big\| \int_0^\delta e^{-it\Delta} P_{\geq N} F(u(t))\, dt \Big\|_{L_x^2} \leq N^{-s} + \tfrac{1}{10} \sum_{M \leq \eta N} \big(\tfrac{M}{N}\big)^s \mathcal{M}(M),$$

provided N is sufficiently large depending on u, s, and η. An analogous estimate holds for integration over $[-\delta, 0]$ and after pre-multiplication by $\chi_N P^{\pm}$.

Proof. By Lemma 2.7, it suffices to prove

$$\mathcal{N}(N) := \|P_{\geq N} F(u)\|_{L^{\frac{2(d+2)}{d+4}}(J \times \mathbb{R}^d)} \lesssim_u N^{-s-\varepsilon} + \sum_{M \leq \eta N} \big(\tfrac{M}{N}\big)^{s+\varepsilon} \mathcal{M}(M) \tag{63}$$

for some small $\varepsilon > 0$, any interval J of length $|J| \leq \delta$, and all sufficiently large N depending on u, s, and η, since the claim would follow by requiring η small and N large, both depending on u.

From (58), there exists $N_0 = N_0(u, \eta)$ such that

$$\|u_{\geq N_0}\|_{L_t^\infty L_x^2(\mathbb{R} \times \mathbb{R}^d)} \leq \eta^{100 d^2}. \tag{64}$$

Let $N > N_1 := \eta^{-1} N_0$. We decompose

$$\begin{aligned} F(u) = F(u_{\leq \eta N}) &+ O\big(|u_{\leq N_0}|^{\frac{4}{d}} |u_{>\eta N}|\big) + O\big(|u_{N_0 \leq \cdot \leq \eta N}|^{\frac{4}{d}} |u_{>\eta N}|\big) \\ &+ O\big(|u_{>\eta N}|^{1+\frac{4}{d}}\big). \end{aligned} \tag{65}$$

Using Lemma 2.2, Corollary 2.5 together with (53), and Lemma 2.7, we estimate the contribution of the first term on the right-hand side of (65) as follows:

$$\begin{aligned} \|P_{\geq N} F(u_{\leq \eta N})\|_{L^{\frac{2(d+2)}{d+4}}(J \times \mathbb{R}^d)} &\lesssim N^{-s-3\varepsilon} \big\| |\nabla|^{s+3\varepsilon} F(u_{\leq \eta N}) \big\|_{L^{\frac{2(d+2)}{d+4}}(J \times \mathbb{R}^d)} \\ &\lesssim_u \langle \delta \rangle^{\frac{2}{d+2}} N^{-s-3\varepsilon} \big\| |\nabla|^{s+3\varepsilon} u_{\leq \eta N} \big\|_{L_{t,x}^{\frac{2(d+2)}{d}}(J \times \mathbb{R}^d)} \\ &\lesssim_u \langle \delta \rangle^{\frac{2}{d+2}} \sum_{M \leq \eta N} \big(\tfrac{M}{N}\big)^{s+3\varepsilon} [\mathcal{M}(M) + \mathcal{N}(M)] \\ &\lesssim_u \eta^\varepsilon \langle \delta \rangle^{\frac{2}{d+2}} \sum_{M \leq \eta N} \big(\tfrac{M}{N}\big)^{s+2\varepsilon} [\mathcal{M}(M) + \mathcal{N}(M)], \end{aligned}$$

for any $0 < \varepsilon < \frac{1}{3}(1 + \frac{4}{d} - s)$.

To estimate the contribution of the second term on the right-hand side of (65), we use Hölder's inequality, Lemma 2.2, and (53):

$$\begin{aligned} &\big\| O\big(|u_{\leq N_0}|^{\frac{4}{d}} |u_{>\eta N}|\big) \big\|_{L^{\frac{2(d+2)}{d+4}}(J \times \mathbb{R}^d)} \\ &\quad \lesssim \delta^{\frac{1}{2}} \|u_{\leq N_0}\|^{\frac{2}{d}}_{L_{t,x}^{\frac{2(d+2)}{d}}(J \times \mathbb{R}^d)} \|u_{\leq N_0}\|^{\frac{2}{d}}_{L^\infty_{t,x}(J \times \mathbb{R}^d)} \|u_{>\eta N}\|_{L_t^\infty L_x^2(J \times \mathbb{R}^d)} \\ &\quad \lesssim_u \delta^{\frac{1}{2}} \langle \delta \rangle^{\frac{1}{d+2}} N_0 \mathcal{M}(\eta N). \end{aligned}$$

Finally, to estimate the contribution of the last two terms on the right-hand side of (65), we use Hölder's inequality, interpolation combined with (55) and (64), and then Lemma 2.7 to obtain

$$\begin{aligned}
&\big\|O\big(|u_{N_0\leq\cdot\leq\eta N}|^{\frac4d}|u_{>\eta N}|\big)\big\|_{L^{\frac{2(d+2)}{d+4}}(J\times\mathbb{R}^d)}\\
&\quad\lesssim \|u_{N_0\leq\cdot\leq\eta N}\|^{\frac4d}_{L_{t,x}^{\frac{2(d+2)}{d}}(J\times\mathbb{R}^d)}\|u_{>\eta N}\|_{L_{t,x}^{\frac{2(d+2)}{d}}(J\times\mathbb{R}^d)}\\
&\quad\lesssim \|u_{N_0\leq\cdot\leq\eta N}\|^{\frac{8}{d(d+2)}}_{L_t^\infty L_x^2(J\times\mathbb{R}^d)}\|u_{N_0\leq\cdot\leq\eta N}\|^{\frac{4}{d+2}}_{L_t^2 L_x^{\frac{2d}{d-2}}(J\times\mathbb{R}^d)}[\mathcal{M}(\eta N)+\mathcal{N}(\eta N)]\\
&\quad\lesssim_u \eta^8\langle\delta\rangle^{\frac{2}{d+2}}[\mathcal{M}(\eta N)+\mathcal{N}(\eta N)]
\end{aligned}$$

and similarly,

$$\big\|O\big(|u_{>\eta N}|^{1+\frac4d}\big)\big\|_{L^{\frac{2(d+2)}{d+4}}(J\times\mathbb{R}^d)}\lesssim_u \eta^8\langle\delta\rangle^{\frac{2}{d+2}}[\mathcal{M}(\eta N)+\mathcal{N}(\eta N)].$$

Putting everything together and taking η sufficiently small depending on u and s, then δ sufficiently small depending upon N_0 and η, we derive

$$\mathcal{N}(N)\leq\sum_{M\leq\eta N}\big(\tfrac{M}{N}\big)^{s+2\varepsilon}[\mathcal{M}(M)+\mathcal{N}(M)] \tag{66}$$

for all $N>N_1$ and some (very small) $\varepsilon>0$. The claim (63) follows from this and Lemma 2.1. More precisely, let $\eta=2^{-K}$ where K is sufficiently large so that $2\log(K-1)<\varepsilon(K-1)$. If we write $r=2^{-s-2\varepsilon}$, $x_k=\mathcal{N}(2^kN_1)$, and

$$\begin{aligned}
b_k&=\sum_{l\leq k-K}2^{-(s+2\varepsilon)(k-l)}\mathcal{M}(2^lN_1)+\sum_{l\leq -1}2^{-(s+2\varepsilon)(k-l)}\mathcal{N}(2^lN_1)\\
&\lesssim_u\sum_{l\leq k-K}2^{-(s+2\varepsilon)(k-l)}\mathcal{M}(2^lN_1)+2^{-(s+2\varepsilon)k},
\end{aligned}$$

then (66) implies (16). With a few elementary manipulations, (17) implies (63).

The last claim follows from Lemma 4.2 after employing $P_{\geq N}=P_{\geq N/2}P_{\geq N}$. □

To estimate the integrals where $|t|\geq\delta$, we break the region of (t,y) integration into two pieces, namely, where $|y|\gtrsim M|t|$ and $|y|\ll M|t|$. The former is the more significant region; it contains the points where the integral kernels $P_Me^{-it\Delta}(x,y)$ and $P_M^{\pm}e^{-it\Delta}(x,y)$ are large (see Lemmas 2.6 and 4.1). More precisely, when $|x|\leq N^{-1}$, we use (61); in this case $|y-x|\sim M|t|$ implies $|y|\gtrsim M|t|$ for $|t|\geq\delta\geq N^{-2}$. (This last condition can be subsumed under our hypothesis N sufficiently large depending on u and η.) When $|x|\geq N^{-1}$, we use (60); in this case $|y|-|x|\sim M|t|$ implies $|y|\gtrsim M|t|$.

The next lemma bounds the integrals over the significant region $|y|\gtrsim M|t|$. Let χ_k denote the characteristic function of the set

$$\{(t,y):\,2^k\delta\leq|t|\leq 2^{k+1}\delta,\ |y|\gtrsim M|t|\}.$$

Lemma 5.4 (Main contribution). *Let $0 < s < 1 + \frac{4}{d}$, let $\eta > 0$ be a small number, and let δ be as in Lemma 5.3. Then*

$$\sum_{M \geq N} \sum_{k=0}^{\infty} \Big\| \int_{\mathbb{R}} \int_{\mathbb{R}^d} [P_M e^{-it\Delta}](x,y)\, \chi_k(t,y)\, [\tilde{P}_M F(u(t))](y)\, dy\, dt \Big\|_{L_x^2} \leq \tfrac{1}{10} \sum_{L \leq \eta N} \big(\tfrac{L}{N}\big)^s \mathcal{M}(L)$$

for all N sufficiently large depending on u, s, and η. An analogous estimate holds with P_M replaced by $\chi_N P_M^+$ or $\chi_N P_M^-$; moreover, the time integrals may be taken over $[-2^{k+1}\delta, -2^k\delta]$.

Proof. We decompose

$$F(u) = F(u_{\leq \eta M}) + O\big(|u_{>\eta M}|^{1+\frac{4}{d}}\big) + O\big(|u_{\leq \eta M}|^{\frac{4}{d}} |u_{>\eta M}|\big). \tag{67}$$

We first consider the contribution coming from the last two terms in the decomposition above. By the adjoint Strichartz inequality and Hölder's inequality,

$$\begin{aligned}
&\Big\| \int_{\mathbb{R}} \int_{\mathbb{R}^d} [P_M e^{-it\Delta}](x,y) \chi_k(t,y) \tilde{P}_M \big[O\big(|u_{>\eta M}|^{\frac{d+4}{d}}\big) + O\big(|u_{\leq \eta M}|^{\frac{4}{d}} |u_{>\eta M}|\big)\big](y)\, dy\, dt \Big\|_{L_x^2} \\
&\quad \lesssim \big\| \chi_k \tilde{P}_M \big[O\big(|u_{>\eta M}|^{\frac{d+4}{d}}\big) + O\big(|u_{\leq \eta M}|^{\frac{4}{d}} |u_{>\eta M}|\big)\big] \big\|_{L_t^1 L_y^2} \\
&\quad \lesssim (M 2^k \delta)^{-\frac{2(d-1)}{d}} (2^k\delta)^{\frac{d-1}{d}} \Big[\big\| |y|^{\frac{2(d-1)}{d}} \tilde{P}_M O\big(|u_{>\eta M}|^{\frac{d+4}{d}}\big) \big\|_{L_t^d L_y^2([2^k\delta, 2^{k+1}\delta] \times \mathbb{R}^d)} \\
&\qquad\qquad + \big\| |y|^{\frac{2(d-1)}{d}} \tilde{P}_M O\big(|u_{\leq \eta M}|^{\frac{4}{d}} |u_{>\eta M}|\big) \big\|_{L_t^d L_y^2([2^k\delta, 2^{k+1}\delta] \times \mathbb{R}^d)} \Big].
\end{aligned}$$

As $\tilde{P}_M$ is a Mihlin multiplier and $|y|^{\frac{4(d-1)}{d}}$ is an A_2 weight, $\tilde{P}_M$ is bounded on $L^2(|y|^{\frac{4(d-1)}{d}}\, dy)$; see [34, Ch. V]. Thus, by Hölder's inequality and (56),

$$\begin{aligned}
&\Big\| \int_{\mathbb{R}} \int_{\mathbb{R}^d} [P_M e^{-it\Delta}](x,y) \chi_k(t,y) \tilde{P}_M \big[O\big(|u_{>\eta M}|^{\frac{d+4}{d}}\big) + O\big(|u_{\leq \eta M}|^{\frac{4}{d}} |u_{>\eta M}|\big)\big](y)\, dy\, dt \Big\|_{L_x^2} \\
&\quad \lesssim (M 2^k \delta)^{-\frac{2(d-1)}{d}} (2^k\delta)^{\frac{d-1}{d}} \Big[\big\| |y|^{\frac{2(d-1)}{d}} |u_{>\eta M}|^{\frac{d+4}{d}} \big\|_{L_t^d L_y^2([2^k\delta, 2^{k+1}\delta] \times \mathbb{R}^d)} \\
&\qquad\qquad + \big\| |y|^{\frac{2(d-1)}{d}} |u_{\leq \eta M}|^{\frac{4}{d}} |u_{>\eta M}| \big\|_{L_t^d L_y^2([2^k\delta, 2^{k+1}\delta] \times \mathbb{R}^d)} \Big] \\
&\quad \lesssim (M 2^k \delta)^{-\frac{2(d-1)}{d}} (2^k\delta)^{\frac{d-1}{d}} \|u_{>\eta M}\|_{L_t^\infty L_y^2} \Big[\big\| |y|^{\frac{d-1}{2}} u_{>\eta M} \big\|_{L_t^4 L_y^\infty([2^k\delta, 2^{k+1}\delta] \times \mathbb{R}^d)}^{\frac{4}{d}} \\
&\qquad\qquad + \big\| |y|^{\frac{d-1}{2}} u_{\leq \eta M} \big\|_{L_t^4 L_y^\infty([2^k\delta, 2^{k+1}\delta] \times \mathbb{R}^d)}^{\frac{4}{d}} \Big] \\
&\quad \lesssim_u (M 2^k \delta)^{-\frac{2(d-1)}{d}} (2^k\delta)^{\frac{d-1}{d}} \mathcal{M}(\eta N) \langle 2^k \delta \rangle^{\frac{1}{d}}.
\end{aligned}$$

Summing first in $k \geq 0$ and then in $M \geq N$, we estimate the contribution of the last two terms on the right-hand side of (67) by

$$\big(N^2\delta\big)^{-1+\frac{1}{d}} \mathcal{M}(\eta N).$$

Next we consider the contribution coming from the first term on the right-hand side of (67). By the adjoint of the weighted Strichartz inequality in Lemma 2.8,

Hölder's inequality, Corollary 2.5, and Lemma 2.2,

$$\begin{aligned}
\Big\| \int_{\mathbb{R}}\int_{\mathbb{R}^d} &[P_M e^{-it\Delta}](x,y)\, \chi_k(t,y)\, [\tilde{P}_M F(u_{\leq \eta M}(t))](y)\, dy\, dt \Big\|_{L^2_x} \\
&\lesssim (M2^k\delta)^{-\frac{2(d-1)}{q}} \big\| \chi_k \tilde{P}_M F(u_{\leq \eta M}) \big\|_{L_t^{\frac{q}{q-1}} L_y^{\frac{2q}{q+4}}} \\
&\lesssim (M2^k\delta)^{-\frac{2(d-1)}{q}} (2^k\delta)^{\frac{q-1}{q}} M^{-s} \big\| |\nabla|^s F(u_{\leq \eta M}) \big\|_{L_t^\infty L_y^{\frac{2q}{q+4}}} \\
&\lesssim (M2^k\delta)^{-\frac{2(d-1)}{q}} (2^k\delta)^{\frac{q-1}{q}} M^{-s} \|u_{\leq \eta M}\|^{\frac{4}{d}}_{L_t^\infty L_y^{\frac{2q}{d}}} \big\| |\nabla|^s u_{\leq \eta M} \big\|_{L_t^\infty L_y^2} \\
&\lesssim_u (M2^k\delta)^{-\frac{2(d-1)}{q}} (2^k\delta)^{\frac{q-1}{q}} (\eta M)^{\frac{2(q-d)}{q}} \sum_{L \leq \eta M} \big(\tfrac{L}{M}\big)^s \mathcal{M}(L) \\
&\lesssim_u (M^2 2^k \delta)^{-\frac{2d-q-1}{q}} \sum_{L\leq \eta N} \big(\tfrac{L}{N}\big)^s \mathcal{M}(L)
\end{aligned}$$

provided $q \geq \max\{d, 4\}$ and $M \geq N$. In order to deduce the last inequality, we used the fact that for $M \geq N$,

$$\begin{aligned}
\sum_{L\leq \eta M} \big(\tfrac{L}{M}\big)^s \mathcal{M}(L) &\leq \sum_{L\leq \eta N} \big(\tfrac{L}{N}\big)^s \mathcal{M}(L) + \sum_{\eta N \leq L \leq \eta M} \big(\tfrac{L}{M}\big)^s \mathcal{M}(L) \\
&\lesssim \sum_{L\leq \eta N} \big(\tfrac{L}{N}\big)^s \mathcal{M}(L) + \eta^s \mathcal{M}(\eta N) \\
&\lesssim \sum_{L\leq \eta N} \big(\tfrac{L}{N}\big)^s \mathcal{M}(L).
\end{aligned} \tag{68}$$

Therefore, choosing $q = d+1$,

$$\begin{aligned}
\sum_{M\geq N} \sum_{k=0}^{\infty} \Big\| \int_{\mathbb{R}}\int_{\mathbb{R}^d} &[P_M e^{-it\Delta}](x,y)\, \chi_k(t,y)\, [\tilde{P}_M F(u_{\leq \eta M}(t))](y)\, dy\, dt \Big\|_{L^2_x} \\
&\lesssim (N^2\delta)^{-\frac{d-2}{d+1}} \sum_{L\leq \eta N} \big(\tfrac{L}{N}\big)^s \mathcal{M}(L).
\end{aligned}$$

Putting everything together we obtain

$$\begin{aligned}
\sum_{M\geq N} \sum_{k=0}^{\infty} \Big\| \int_{\mathbb{R}}\int_{\mathbb{R}^d} &[P_M e^{-it\Delta}](x,y)\, \chi_k(t,y)\, [\tilde{P}_M F(u(t))](y)\, dy\, dt \Big\|_{L^2_x} \\
&\lesssim_u \eta^{-s} \Big[\big(N^2\delta\big)^{-1+\frac{2}{d}} + \big(N^2\delta\big)^{-1+\frac{1}{d}} + \big(N^2\delta\big)^{-\frac{d-2}{d+1}} \Big] \sum_{L\leq \eta N} \big(\tfrac{L}{N}\big)^s \mathcal{M}(L).
\end{aligned}$$

Choosing N sufficiently large depending on u, δ, and s (and hence only on u, η, and s), we obtain the desired bound.

The last claim follows from the L^2_x-boundedness of $\chi_N P^{\pm} P_M$ (cf. Lemma 4.2) and the time-reversal symmetry of the argument just presented. □

We turn now to the region of (t, y) integration where $|y| \ll M|t|$. First, we describe the bounds that we will use for the kernels of the propagators. For $|x| \leq N^{-1}$, $|y| \ll M|t|$, and $|t| \geq \delta \gg N^{-2}$,

$$|P_M e^{-it\Delta}(x,y)| \lesssim \frac{1}{(M^2|t|)^{50d}} \frac{M^d}{\langle M(x-y)\rangle^{50d}}; \tag{69}$$

this follows from Lemma 2.6 since under these constraints, $|y-x| \ll M|t|$. For $|x| \geq N^{-1}$ and y and t as above,

$$|P_M^{\pm} e^{-it\Delta}(x,y)| \lesssim \frac{1}{(M^2|t|)^{50d}} \frac{M^d}{\langle Mx\rangle^{\frac{d-1}{2}} \langle My\rangle^{\frac{d-1}{2}} \langle M|x| - M|y|\rangle^{50d}}; \tag{70}$$

by Lemma 4.1. Note that we have used $|y| - |x| \ll M|t|$ and

$$\langle M^2|t| + M|x| - M|y|\rangle^{-100d} \lesssim (M^2|t|)^{-50d} \langle M|x| - M|y|\rangle^{-50d}$$

in order to simplify the bound.

From (69) and (70) we see that under the hypotheses set out above,

$$|P_M e^{-it\Delta}(x,y)| + |P_M^{\pm} e^{-it\Delta}(x,y)| \lesssim \frac{1}{(M^2|t|)^{50d}} K_M(x,y), \tag{71}$$

where

$$K_M(x,y) := \frac{M^d}{\langle M(x-y)\rangle^{50d}} + \frac{M^d}{\langle Mx\rangle^{\frac{d-1}{2}} \langle My\rangle^{\frac{d-1}{2}} \langle M|x| - M|y|\rangle^{50d}}.$$

Note that by Schur's test, this is the kernel of a bounded operator on $L_x^2(\mathbb{R}^d)$.

Let $\tilde{\chi}_k$ denote the characteristic function of the set

$$\{(t,y) : \, 2^k\delta \leq |t| \leq 2^{k+1}\delta, \ |y| \ll M|t|\}.$$

Lemma 5.5 (The tail)**.** *Let $0 < s < 1 + \frac{4}{d}$, let $\eta > 0$ be a small number, and let δ be as in Lemma 5.3. Then*

$$\sum_{M \geq N} \sum_{k=0}^{\infty} \Big\| \int_{\mathbb{R}} \int_{\mathbb{R}^d} \frac{K_M(x,y)}{(M^2|t|)^{50d}} \, \tilde{\chi}_k(t,y) \, |\tilde{P}_M F(u(t))|(y) \, dy \, dt \Big\|_{L_x^2} \leq \tfrac{1}{10} \sum_{L \leq \eta N} \big(\tfrac{L}{N}\big)^s \mathcal{M}(L)$$

for all N sufficiently large depending on u, s, and η (in particular, we require $N \gg \delta^{-1/2}$).

Proof. Using Hölder's inequality, the L^2-boundedness of the operator with kernel K_M, and Lemma 2.2,

$$\begin{aligned}
&\Big\| \int_{\mathbb{R}} \int_{\mathbb{R}^d} \frac{K_M(x,y)}{(M^2|t|)^{50d}} \, \tilde{\chi}_k(t,y) \, |\tilde{P}_M F(u(t))|(y) \, dy \, dt \Big\|_{L_x^2} \\
&\qquad \lesssim (M^2 2^k \delta)^{-50d} (2^k\delta)^{\frac{d-2}{d}} M^{\frac{2(d-2)}{d}} \|\tilde{P}_M F(u)\|_{L_t^{\frac{d}{2}} L_x^{\frac{2d^2}{d^2+4d-8}}([2^k\delta, 2^{k+1}\delta] \times \mathbb{R}^d)} \\
&\qquad \lesssim (M^2 2^k \delta)^{-49d} \|\tilde{P}_M F(u)\|_{L_t^{\frac{d}{2}} L_x^{\frac{2d^2}{d^2+4d-8}}([2^k\delta, 2^{k+1}\delta] \times \mathbb{R}^d)}.
\end{aligned}$$

We decompose

$$F(u) = F(u_{\leq \eta M}) + O\big(|u_{\leq \eta M}|^{\frac{4}{d}} |u_{> \eta M}|\big) + O\big(|u_{> \eta M}|^{1+\frac{4}{d}}\big) \tag{72}$$

Discarding the projection $\tilde P_M$, we use Hölder and (55) to estimate

$$\begin{aligned}\big\|\tilde P_M O\big(|u_{\leq \eta M}|^{\frac4d}|u_{>\eta M}|\big)\big\|_{L_t^{\frac d2}L_x^{\frac{2d^2}{d^2+4d-8}}([2^k\delta,2^{k+1}\delta]\times\mathbb{R}^d)} &\lesssim \|u_{\leq \eta M}\|^{\frac4d}_{L_t^2L_x^{\frac{2d}{d-2}}([2^k\delta,2^{k+1}\delta]\times\mathbb{R}^d)}\|u_{>\eta M}\|_{L_t^\infty L_x^2}\\ &\lesssim_u \langle 2^k\delta\rangle^{\frac2d}\mathcal{M}(\eta N)\\ \big\|\tilde P_M O\big(|u_{>\eta M}|^{1+\frac4d}\big)\big\|_{L_t^{\frac d2}L_x^{\frac{2d^2}{d^2+4d-8}}([2^k\delta,2^{k+1}\delta]\times\mathbb{R}^d)} &\lesssim \|u_{>\eta M}\|^{\frac4d}_{L_t^2L_x^{\frac{2d}{d-2}}([2^k\delta,2^{k+1}\delta]\times\mathbb{R}^d)}\|u_{>\eta M}\|_{L_t^\infty L_x^2}\\ &\lesssim_u \langle 2^k\delta\rangle^{\frac2d}\mathcal{M}(\eta N).\end{aligned}$$

To estimate the contribution coming from the first term on the right-hand side of (72), we use Lemma 2.2, Corollary 2.5 (with $r=\frac{d^2}{d-2}$) combined with Hölder's inequality in the time variable, (55), and (68), to estimate

$$\begin{aligned}\|\tilde P_M F(u_{\leq\eta M})\|_{L_t^{\frac d2}L_x^{\frac{2d^2}{d^2+4d-8}}([2^k\delta,2^{k+1}\delta]\times\mathbb{R}^d)} &\lesssim M^{-s}\big\||\nabla|^sF(u_{\leq \eta M})\big\|_{L_t^{\frac d2}L_x^{\frac{2d^2}{d^2+4d-8}}([2^k\delta,2^{k+1}\delta]\times\mathbb{R}^d)}\\ &\lesssim M^{-s}\big\||\nabla|^s u_{\leq\eta M}\big\|_{L_t^\infty L_x^2}\|u_{\leq \eta M}\|^{\frac4d}_{L_t^2L_x^{\frac{2d}{d-2}}([2^k\delta,2^{k+1}\delta]\times\mathbb{R}^d)}\\ &\lesssim_u \langle 2^k\delta\rangle^{\frac2d}\sum_{L\leq \eta M}\big(\tfrac LM\big)^s\mathcal{M}(L)\\ &\lesssim_u \langle 2^k\delta\rangle^{\frac2d}\sum_{L\leq \eta N}\big(\tfrac LN\big)^s\mathcal{M}(L)\end{aligned}$$

for any $M\geq N$.

Putting everything together, we deduce

$$\begin{aligned}\Big\|\int_{\mathbb{R}}\int_{\mathbb{R}^d}\frac{K_M(x,y)}{(M^2|t|)^{50d}}\,\tilde\chi_k(t,y)\,[\tilde P_M F(u(t))](y)\,dy\,dt\Big\|_{L_x^2}&\\ \lesssim_u (M^22^k\delta)^{-49d}\langle 2^k\delta\rangle^{\frac2d}\eta^{-s}&\sum_{L\leq\eta N}\big(\tfrac LN\big)^s\mathcal{M}(L).\end{aligned}$$

Summing over $k\geq 0$ and $M\geq N$, we obtain

$$\begin{aligned}\sum_{M\geq N}\sum_{k=0}^\infty\Big\|\int_{\mathbb{R}}\int_{\mathbb{R}^d}\frac{K_M(x,y)}{(M^2|t|)^{50d}}\,\tilde\chi_k(t,y)\,[\tilde P_M F(u(t))](y)\,dy\,dt\Big\|_{L_x^2}&\\ \lesssim_u (N^2\delta)^{-49d}\eta^{-s}&\sum_{L\leq\eta N}\big(\tfrac LN\big)^s\mathcal{M}(L).\end{aligned}$$

The claim follows by choosing N sufficiently large depending on δ, η, and s (and hence only on u, s, and η). □

We have now gathered enough information to complete the

Proof of Proposition 5.2. Naturally, we may bound $\|u_{\geq N}\|_{L^2}$ by separately bounding the L^2 norm on the ball $\{|x|\leq N^{-1}\}$ and on its complement. On the ball, we use (61), while outside the ball we use (60). Invoking (62) and the triangle

inequality, we reduce the proof to bounding certain integrals. The integrals over short times were estimated in Lemma 5.3. For $|t| \geq \delta$, we further partition the region of integration into two pieces. The first piece, where $|y| \gtrsim M|t|$, was dealt with in Lemma 5.4. To estimate the remaining piece, $|y| \ll M|t|$, one combines (71) and Lemma 5.5. $\square$

6. The double high-to-low frequency cascade

In this section, we use the additional regularity provided by Theorem 5.1 to preclude double high-to-low frequency cascade solutions. We argue as in [26].

Proposition 6.1 (Absence of double cascades)**.** *Let $d \geq 3$. There are no non-zero global spherically symmetric solutions to* (1) *that are double high-to-low frequency cascades in the sense of Theorem 1.10.*

Proof. Suppose to the contrary that there is such a solution u. By Theorem 5.1, u lies in $C_t^0 H_x^1(\mathbb{R} \times \mathbb{R}^d)$. Hence the energy

$$E(u) = E(u(t)) = \int_{\mathbb{R}^d} \tfrac{1}{2}|\nabla u(t,x)|^2 + \mu \tfrac{d}{2(d+2)}|u(t,x)|^{2(d+2)/d}\,dx$$

is finite and conserved (see e.g. [10]). As we have $M(u) < M(Q)$ in the focusing case, the sharp Gagliardo-Nirenberg inequality (reproduced here as Theorem 1.6) gives

$$\|\nabla u(t)\|_{L_x^2(\mathbb{R}^d)}^2 \sim_u E(u) \sim_u 1 \tag{73}$$

for all $t \in \mathbb{R}$. We will now reach a contradiction by proving that $\|\nabla u(t)\|_2 \to 0$ along any sequence where $N(t) \to 0$. The existence of two such time sequences is guaranteed by the fact that u is a double high-to-low frequency cascade.

Let $\eta > 0$ be arbitrary. By Definition 1.8, we can find $C = C(\eta, u) > 0$ such that

$$\int_{|\xi| \geq CN(t)} |\hat{u}(t,\xi)|^2\,d\xi \leq \eta^2$$

for all t. Meanwhile, by Theorem 5.1, $u \in C_t^0 H_x^s(\mathbb{R} \times \mathbb{R}^d)$ for some $s > 1$. Thus,

$$\int_{|\xi| \geq CN(t)} |\xi|^{2s}|\hat{u}(t,\xi)|^2\,d\xi \lesssim_u 1$$

for all t and some $s > 1$. Thus, by Hölder's inequality,

$$\int_{|\xi| \geq CN(t)} |\xi|^2|\hat{u}(t,\xi)|^2\,d\xi \lesssim_u \eta^{2(s-1)/s}.$$

On the other hand, from mass conservation and Plancherel's theorem we have

$$\int_{|\xi| \leq CN(t)} |\xi|^2|\hat{u}(t,\xi)|^2\,d\xi \lesssim_u C^2 N(t)^2.$$

Summing these last two bounds and using Plancherel's theorem again, we obtain

$$\|\nabla u(t)\|_{L_x^2(\mathbb{R}^d)} \lesssim_u \eta^{(s-1)/s} + CN(t)$$

for all t. As $\eta > 0$ is arbitrary and there exists a sequence of times $t_n \to \infty$ such that $N(t_n) \to 0$ (u is a double high-to-low frequency cascade), we conclude $\|\nabla u(t_n)\|_2 \to 0$. This contradicts (73). $\square$

Remark. As mentioned in [26], the argument presented can be used to rule out non-radial single-sided cascade solutions that lie in $C_t^0 H_x^s$ for some $s > 1$. (By a single-sided cascade we mean a solution with $N(t)$ bounded on a semi-infinite interval, say $[T, \infty)$, with $\liminf_{t\to\infty} N(t) = 0$.) For such regular solutions u, we may define the total momentum $\int_{\mathbb{R}^d} \operatorname{Im}(\overline{u}\nabla u)$, which is conserved. By a Galilean transformation, we can set this momentum equal to zero; thus $\int_{\mathbb{R}^d} \xi |\hat{u}(t,\xi)|^2 \, d\xi = 0$. From this, mass conservation, and the uniform H_x^s bound for some $s > 1$, one can show that $\xi(t) \to 0$ whenever $N(t) \to 0$. On the other hand, a modification of the above argument gives

$$1 \sim_u \|\nabla u(t)\|_2 \lesssim \eta^{(s-1)/s} + C\big(N(t) + |\xi(t)|\big),$$

which is absurd.

7. Death of a soliton

In this section, we use the additional regularity proved in Theorem 5.1 to rule out the third and final enemy, the soliton-like solution. Once again, we follow [26]; the method is similar to that in [23]. Let

$$M_R(t) := 2 \operatorname{Im} \int_{\mathbb{R}^d} \psi(|x|/R) \bar{u}(t,x) x \cdot \nabla u(x,t) \, dx \tag{74}$$

where ψ is a smooth function obeying

$$\psi(r) = \begin{cases} 1 & : r \le 1 \\ 0 & : r \ge 2 \end{cases}$$

and R denotes a radius to be chosen momentarily. For solutions u to (1) belonging to $C_t^0 H_x^1$, $M_R(t)$ is a well-defined function. Indeed,

$$|M_R(t)| \lesssim R \|u(t)\|_2 \|\nabla u(t)\|_2 \lesssim_u R. \tag{75}$$

An oft-repeated calculation (essentially that in the derivation of the Morawetz and viriel identities) gives the following

Lemma 7.1.

$$\begin{aligned} \partial_t M_a(t) = 8E(u(t)) & \\ - \int_{\mathbb{R}^d} \Big[\tfrac{d^2-1}{R|x|} \psi'\big(\tfrac{|x|}{R}\big) + \tfrac{2d+1}{R^2} \psi''\big(\tfrac{|x|}{R}\big) + \tfrac{|x|}{R^3} \psi'''\big(\tfrac{|x|}{R}\big) \Big] |u(t,x)|^2 \, dx & \qquad (76) \\ + 4 \int_{\mathbb{R}^d} \Big[\psi\big(\tfrac{|x|}{R}\big) - 1 + \tfrac{|x|}{R} \psi'\big(\tfrac{|x|}{R}\big) \Big] |\nabla u(t,x)|^2 \, dx & \qquad (77) \\ + \tfrac{4\mu}{d+2} \int_{\mathbb{R}^d} \Big[d\Big(\psi\big(\tfrac{|x|}{R}\big) - 1\Big) + \tfrac{|x|}{R} \psi'\big(\tfrac{|x|}{R}\big) \Big] |u(t,x)|^{\frac{2(d+2)}{d}} \, dx, & \qquad (78) \end{aligned}$$

where $E(u)$ is the energy of u as defined in (7).

Proposition 7.2 (Absence of solitons)**.** *Let $d \ge 3$. There are no non-zero global spherically symmetric solutions to* (1) *that are soliton-like in the sense of Theorem 1.10.*

Proof. Assume to the contrary that there is such a solution u. Then, by Theorem 5.1, $u \in C_t^0 H_x^s$ for some $s > 1$. In particular,

$$|M_R(t)| \lesssim_u R. \tag{79}$$

Recall that in the focusing case, $M(u) < M(Q)$. As a consequence, the sharp Gagliardo–Nirenberg inequality (reproduced here as Theorem 1.6) implies that the energy is a positive quantity in the focusing case as well as in the defocusing case. Indeed,

$$E(u) \gtrsim_u \int_{\mathbb{R}^d} |\nabla u(t,x)|^2\,dx > 0.$$

We will show that (76) through (78) constitute only a small fraction of $E(u)$. Combining this fact with Lemma 7.1, we conclude $\partial_t M_R(t) \gtrsim E(u) > 0$, which contradicts (79).

We first turn our attention to (76). This is trivially bounded by

$$|(76)| \lesssim_u R^{-2}. \tag{80}$$

We now study (77) and (78). Let $\eta > 0$ be a small number to be chosen later. By Definition 1.8 and the fact that $N(t) = 1$ for all $t \in \mathbb{R}$, if R is sufficiently large depending on u and η, then

$$\int_{|x|\geq \frac{R}{4}} |u(t,x)|^2\,dx \leq \eta \tag{81}$$

for all $t \in \mathbb{R}$. Let χ denote a smooth cutoff to the region $|x| \geq \frac{R}{2}$, chosen so that $\nabla\chi$ is bounded by R^{-1} and supported where $|x| \sim R$. As $u \in C_t^0 H_x^s$ for some $s > 1$, using interpolation and (81), we estimate

$$\begin{aligned} |(77)| &\lesssim \|\chi\nabla u(t)\|_2^2 \lesssim \|\nabla(\chi u(t))\|_2^2 + \|u(t)\nabla\chi\|_2^2 \lesssim \|\chi u(t)\|_2^{\frac{2(s-1)}{s}} \|u(t)\|_{H_x^s}^{\frac{2}{s}} + \eta \\ &\lesssim_u \eta^{\frac{s-1}{s}} + \eta. \end{aligned} \tag{82}$$

Finally, we are left to consider (78). Using the same χ as above together with the Gagliardo–Nirenberg inequality and (81),

$$|(78)| \lesssim \|\chi u(t)\|_{\frac{2(d+2)}{d}}^{\frac{2(d+2)}{d}} \lesssim \|\chi u(t)\|_2^{\frac{4}{d}} \|\nabla(\chi u(t))\|_2^2 \lesssim_u \eta^{\frac{2}{d}}. \tag{83}$$

Combining (80), (82), and (83) and choosing η sufficiently small depending on u and R sufficiently large depending on u and η, we obtain

$$|(76)| + |(77)| + |(78)| \leq \tfrac{1}{100} E(u).$$

This completes the proof of the proposition for the reasons explained in the third paragraph. □

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